Exercises March 26th 2004, Optimal Control of Economic Systems

1. Let $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ be given. Derive a curve that connects the two points of minimal length.

2. Consider the criterion:

   $$\int_0^1 y(x)^2 + (\dot{y}(x) - 1)^2 dx, \quad y(0) = 0, \quad y(1) = 1$$

   The problem is to find a function y that minimizes the criterion, subject to the boundary conditions.

   (a) Derive the Euler equation for this problem.

   (b) Determine the solution of the Euler equation.

   (c) Argue that the solution of the Euler equation indeed minimizes the criterion.

3. Consider the problem of optimizing the criterion

   $$\int_0^T F(t, x(t), \dot{x}(t)dt, \quad x(0) = x_0$$

   Notice that this problem differs from that in Section 2.2 in that the final value of x is not restricted. Other than that we adopt the same assumptions. The aim of the exercise is to derive necessary conditions for optimality.

   Prove that if x optimizes the criterion, then

   $$\frac{\partial F}{\partial x}(t, x(t), \dot{x}(t)) - \frac{d}{dt}(\frac{\partial F}{\partial \dot{x}}(t, x(t), \dot{x}(t))) = 0 \quad \text{and} \quad \frac{\partial F}{\partial \dot{x}}(T, x(T), \dot{x}(T)) = 0.$$

   Hint: the second condition may be derived by close inspection of the proof of Theorem 2.2.3.

4. A curve needs be drawn in the plane connecting the points $(x, y) = (0, 2)$ and $(2, 0)$. The curve should reach the zero level as fast as possible, on the other hand the steepness should not be too big. This problem can be modelled as follows: Minimize

   $$\int_0^2 y(x) + \dot{y}(x)^2 dx \quad y(0) = 2 \quad y(2) = 0 \tag{1}$$

   (a) Find a function $y(x)$ that satisfies the necessary conditions for optimality.

   (b) Same problem except that now $y(2)$ is free.