Exercises April 16th 2004, Optimal Control of Economic Systems

1. Consider:

$$\dot{x}_1(t) = -x_1(t)x_2(t) - x_1(t)x_2^2(t)$$
$$\dot{x}_2(t) = x_1^2(t) - x_1^2(t)x_2(t) - x_2(t).$$

   (a) Determine all equilibrium points.

   (b) Prove that the origin is stabel.

   (c) Is the origin asymptotically stable?

2. Investigate the stability of the origin for the following systems. Use a suitable Lyapunov function.

   a.

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_1^3(t). \end{aligned}$$

   b.

$$\begin{aligned} \dot{x}_1(t) &= -x_1^3(t) - x_2^2(t) \\ \dot{x}_2(t) &= x_1(t)x_2(t) - x_2^3(t). \end{aligned}$$

3. Consider the scalar equation

$$\dot{x}(t) = ax^3(t)$$

   with $a \in \mathbb{R}$.

   a. Prove that the linearization of this system about its equilibrium point is independent of a.

   b. Prove that, depending on a, the equilibrium point may be asymptotically stable, stable but not asymptotically stable, and unstable.

4. Consider the system

$$\begin{aligned} \dot{x}_1(t) &= -2x_1(t)\,[x_1(t) - 1]\,[2x_1(t) - 1] \\ \dot{x}_2(t) &= -2x_2(t). \end{aligned} \tag{1}$$

   a. Calculate all equilibrium points of the system (1).

   b. Prove that there are two asymptotically stable equilibrium points.

   c. Investigate the stability of the other equilibrium point(s).