Exercises May 28th 2004, Optimal Control of Economic Systems

1. Consider the discrete time system and cost criterion

$$x(k+1) = Ax(k) + Bu(k) \quad J(x_0, u) = x(N)^T Gx(N) + \sum_{k=0}^{N-1} x(k)^T Qx(k) + u(k)^T Ru(k).$$

   Here $Q = Q^T \geq 0$, $G = G^T \geq 0$, and $R = R^T > 0$. Denote by $V(x, k)$ the optimal cost from time k to N starting in state x. Of course, V is the value function.

   (a) Show that the value function satisfies:

$$V(x, k-1) = \min_{u \in \mathbb{R}^m} \left[x^T Qx + u^T Ru + V(Ax + Bu, k)\right] \quad V(x, N) = x^T Gx$$

   (b) Assume that the value function is quadratic: $V(x, k) = x^T P(k)x$, where $P(k) = P(k)^T$. Show that $P(k)$ satisfies the Riccati Difference equation:

$$P(k-1) = A^T P(k)A - A^T P(k)B(B^T P(k)B + R)^{-1} B^T P(k)A + Q \quad P(N) = G$$

   (c) Determine an expression for the optimal control $u(k)$.

   (d) Argue that for all $0 \leq k \leq N$: $P(k) \geq 0$.

   (e) Solve the optimal control problem for

$$x(k+1) = x(k) + u(k), \quad x(0) = 1 \quad J(1, u) = \sum_{k=0}^{3} x(k)^2 + u(k)^2.$$

2. Consider the system and cost criterion

$$\frac{d}{dt}x = ax + bu \quad , x(0) = 1 \quad J(1, u) = gx(T)^2 + \int_0^T qx(\tau)^2 + ru(\tau)^2 d\tau$$

   (a) Derive the Riccati differential equation (RDE) associated with the problem of minimizing J.

   (b) Verify that the solution of RDE is given by:

$$p(t) = \frac{ar - \tanh\left(\frac{\sqrt{qrb^2 + a^2r^2}(t+c)}{r}\right)\sqrt{qrb^2 + a^2r^2}}{b^2}$$

   Recall that $\tanh(t) = \frac{e^t - e^{-t}}{e^t + e^{-t}}$ so that $\frac{d}{dt}\tanh(t) = 1 - \tanh(t)^2$.

   Take $a = 1$,$b = 1$,$r = 2$,$q = 6$, and $g = 0$.

   (c) Show that $p(t) = 2 - 4\tanh(2t + 2c)$, with $c = -T + \frac{1}{2}\operatorname{arctanh}(\frac{1}{2})$.

   (d) Intuitively, one could expect that for very large T the variations in p become small. Indeed, determine the limit of $p(0)$ as T tends to infinity:

$$\lim_{T \to \infty} p(0) =: p_\infty$$

   (e) Show that p_∞ satisfies the *Algebraic Riccati Equation*, ARE:

$$2p_\infty - \frac{1}{2}p_\infty^2 + 6 = 0$$

(f) For the finite horizon $T < \infty$, the optimal control is given by:

$$u(t) = -\frac{b}{r}p(t)x(t) = -\frac{1}{2}p(t)x(t).$$

Replace $p(t)$ by p_∞ and calculate the corresponding costs for the infinite horizon $T = \infty$. To that end you need to determine the closed-loop gain $a - b\frac{1}{2}p_\infty$. Notice that $a - b\frac{1}{2}p_\infty < 0$. In other words the infinite horizon feedback stabilizes the system.