Exercises June 18th 2004, Optimal Control of Economic Systems

1. Consider the system and cost criterion:

$$\frac{d}{dt}x = u \quad J(x_0, u) = \int_0^1 2x(t)^2 + 2x(t)u(t) + u(t)^2 dt \tag{1}$$

   (a) Give the Hamiltonian, determine the optimal control, and derive the differential equations for the co-state.

   (b) Solve the differential equations for the state and co-state.

   (c) Assume that the value function for this problem is of the form $p(t)x^2$. Derive a differential equation for the function p.

   (d) Use the relation between the value function and the co-state to solve the differential equation for p.

2. The macro economic policy of a country is determined by two decision makers, DM1 (government) and DM2 (central bank). The government determines the growth rate of real public expenditures and the central bank has the growth rate of the nominal money supply as its instrument variable.

   DMi wishes to minimize

$$J^i(g, m) = \int_0^T [\beta e_i m(t) + \gamma e_i g(t) + f_i \bar{\pi}(t)]dt$$

   where $\bar{\pi}(t)$ denotes the expected inflation rate. Assume that $\bar{\pi}(t)$ satisfies

$$\frac{d}{dt}\bar{\pi}(t) = \beta h m(t) + \gamma h g(t) - \beta h \bar{\pi}(t)$$

   with

$$e_i = \frac{b_i\lambda - a_i\delta}{1+\beta\lambda}, f_i = \frac{b_i\lambda + a_i\beta\delta}{1+\beta\lambda}, h = \frac{\lambda\eta}{1+\beta\lambda}$$

   where $\lambda, \delta, \beta, \gamma, \eta$ are positive constants.

   $g(t)$ denotes the growth rate of real public expenditures and $m(t)$ denotes the growth rate of the nominal money supply. $g(t)$ and $m(t)$ satisfy the following constraints:

$$c_1 \leq g(t) \leq d_1 \text{ and } c_2 \leq m(t) \leq d_2.$$

   Assume that the decision makers play a non-cooperative (Nash) game.

   (a) Give the definition of a Nash equilibrium solution (g^*, m^*).

   (b) Give the Hamiltonian for DM1 and show that the optimal policy is given by

$$g^*(t) = \begin{cases} d_1 & \text{if} \quad -\gamma e_1 + \xi^1(t)\gamma h > 0 \\ c_1 & \text{if} \quad -\gamma e_1 + \xi^1(t)\gamma h < 0 \end{cases}$$

   where $\xi^1(t)$, the co-state variable, is given by

$$\xi^1(t) = \frac{f_1}{\beta h}[e^{-\beta h(T-t)} - 1].$$

   Note that this implies that $\xi^1(t) \leq 0$, $t \in [0, T]$.

   (c) Show that $\xi^1(t)$ increases monotonically towards 0 for $t \to T$. Explain why $g^*(t)$ can only switch from c_1 to d_1 (at most once). Note that a switch can only occur if $e_1 < 0$, i.e. if $a_1\delta > b_1\lambda$.

   (d) Determine the optimal policy $m^*(t)$