Simplest problem of the calculus of variations with end cost

We consider a variation on the problem stated in Section 2.2. The value of the state at time T is not prescribed, rather there is a penalty on the final state. For notational convenience we denote the state by q instead of x.
The problem is as follows:
Determine a function $q(t)$, defined on the interval $[0,T]$, such that the integral

$$J(q(\cdot)) = S(q(T)) + \int_0^T F(t, q(t), \dot{q}(t))dt \tag{1}$$

is maximized (or minimized), and where $q(t)$ in addition satisfies the boundary condition

$$q(0) = q_0 \tag{2}$$

for given q_0.
The following result is similar to Theorem 2.2.3. The difference is that the condition $x(t) = x_T$ is now replaced by a penalty $S(x(T))$.

1 Theorem
Consider the simplest problem in the calculus of variations and suppose Assumptions 2.2.1 and 2.2.2 are met. Then a necessary condition that a C^2 function $q^*(t)$ maximizes (1) and satisfies (2) is that $q^*(t)$ is a solution of the differential equation

$$\frac{\partial F}{\partial q}(t, q(t), \dot{q}(t)) - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}}(t, q(t), \dot{q}(t))\right) = 0 \tag{3}$$

$$\frac{dS}{dq}(q(T)) + \frac{\partial F}{\partial \dot{q}}(T, q^*(T), \dot{q}^*(T)) = 0 \tag{4}$$

Proof Suppose the C^2-function $q^*(t)$ is an optimal solution and let $\delta q(t)$ be any C^2-function on $[0,T]$ that satisfies

$$\delta q(0) = 0 \tag{5}$$

Let $\alpha \in \mathbb{R}$, and define a new C^2-function

$$q(t) = q^*(t) + \alpha \delta q(t) \tag{6}$$

Note that $q(0) = q^*(0) = q_0$, thus $q(t)$ fulfills (2).
Since $q^*(\cdot)$ was supposed to be an optimal solution for our problem we have that

$$J(q^*(\cdot)) \geq J(q^*(\cdot) + \alpha \delta q(\cdot)) \tag{7}$$

for all $\alpha \in \mathbb{R}$. Now suppose the perturbation $\delta q(\cdot)$ in (7) is fixed. Then $J(q^*(\cdot) + \alpha \delta q(\cdot))$ becomes a function of the scalar variable α; say

$$\bar{J}(\alpha) := J(q^*(\cdot) + \alpha \delta q(\cdot)) \tag{8}$$

The optimality condition (7) for the given perturbation $\delta q(\cdot)$ thus translates into the condition

$$\bar{J}(0) \geq \bar{J}(\alpha) \tag{9}$$

for all $\alpha \in \mathbb{R}$, or $\bar{J}$ has a maximum in $\alpha = 0$. Now, given Assumptions 2.2.1 and 2.2.2 and the fact that $\delta q(\cdot)$ is a C^2-function, it follows that the function $\bar{J}(\alpha)$ is differentiable. Therefore, it is clear that at $\alpha = 0$, $\bar{J}'(0) = 0$ since by (9) $\bar{J}$ has a maximum at $\alpha = 0$. Now

$$\begin{aligned}
\bar{J}'(0) &= \frac{d}{d\alpha}\left[S(q(T) + \alpha\delta q(T)) + \int_0^T F(t, q^*(t) + \alpha\delta q(t), \dot{q}^*(t) + \alpha\dot{\delta} q(t))dt\right]_{\alpha=0} \\
&= \frac{dS}{dq}(q(T))\delta q(T) + \int_0^T \left[\frac{\partial F}{\partial q}(t, q^*(t), \dot{q}^*(t))\delta q(t) + \frac{\partial F}{\partial \dot{q}}(t, q^*(t), \dot{q}^*(t))\dot{\delta} q(t)\right] dt \quad (10)
\end{aligned}$$

Integration by parts of the third term in (10) yields

$$\begin{aligned}
&\int_0^T \frac{\partial F}{\partial \dot q}(t, q^*(t), \dot q^*(t))\dot{\delta} q(t)dt = \\
&\frac{\partial F}{\partial \dot q}(t, q^*(t), \dot q^*(t))\delta q(t)\Big|_0^T - \int_0^T \frac{d}{dt}\left(\frac{\partial F}{\partial \dot q}(t, q^*(t), \dot q^*(t))\right)\delta q(t)dt \\
= \quad &\frac{\partial F}{\partial \dot q}(T, q^*(T), \dot q^*(T))\delta q(T) - \int_0^T \frac{d}{dt}\left(\frac{\partial F}{\partial \dot q}(t, q^*(t), \dot q^*(t))\right)\delta q(t)dt. \qquad (11)
\end{aligned}$$

If, in addition to the boundary conditions (5), we also have $\delta q(T) = 0$, then the first terms in (10) and (11) respectively vanish and plugging (11) into (10) yields

$$\begin{aligned}
\bar J'(0) &= \int_0^T \left[\frac{\partial F}{\partial q}(t, q^*(t), \dot q^*(t))\delta q(t) - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot q}(t, q^*(t), \dot q^*(t))\right)\delta q(t)\right]dt \\
&= \int_0^T \left[\frac{\partial F}{\partial q}(t, q^*(t), \dot q^*(t)) - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot q}(t, q^*(t), \dot q^*(t))\right)\right]\delta q(t)dt \\
&= 0 \qquad (12)
\end{aligned}$$

where we also used that $\bar J'(0) = 0$. So far we have assumed that in our derivation the perturbation $\delta q(\cdot)$ was some fixed function. However the equality (12) is obviously true for *all* C^2 perturbations $\delta q(\cdot)$ satisfying (5). But this, implies, via the Lemma 2.2.4, that the term between brackets in (12), i.e.,

$$\frac{\partial F}{\partial q}(t, q^*(t), \dot q^*(t)) - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot q}(t, q^*(t), \dot q^*(t))\right) \qquad (13)$$

vanishes, or in other words (3) is fulfilled.
If, on the other hand we consider a larger class of perturbations δq that only requires $\delta q(0) = 0$, then we get, in addition to (13):

$$\begin{aligned}
\bar J'(0) &= \left[\frac{dS}{dq}(q(T)) + \frac{\partial F}{\partial \dot q}(T, q^*(T), \dot q^*(T))\right]\delta q(T) \qquad (14) \\
&\quad + \int_0^T \left[\frac{\partial F}{\partial q}(t, q^*(t), \dot q^*(t)) - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot q}(t, q^*(t), \dot q^*(t))\right)\right]\delta q(t)dt \\
&= 0. \qquad (15)
\end{aligned}$$

Together with (13) this yields that

$$\frac{dS}{dq}(q(T)) + \frac{\partial F}{\partial \dot q}(T, q^*(T), \dot q^*(T)) = 0 \qquad (16)$$

□

Application to optimal control

This section presents a derivation of the necessary conditions for optimality, the Hamiltonian equations, based on the variational principles discussed in the previous section and Lagrange multipliers. It replaces most of the proofs provided in Chapter 4 of the lecture notes.
All the conditions of Section 4.2 apply.
Consider the cost criterion

$$J(x_0, u) = S(x(T)) + \int_0^T L(x(t), u(t))dt. \qquad (17)$$

The objective is to minimize (17) subject to

$$\frac{d}{dt}x = f(x,u), \quad x(0) = x_0. \tag{18}$$

The following results cover Theorem 4.2.9 and Proposition 4.2.10.

2 Theorem
Suppose $u^*(\cdot) \in \mathcal{U}$ is a solution of the OCP. Then there exists a pair $(x^*(t), \xi^*(t))$ defined on $[0,T]$ such that

$$\dot{x}^* = \frac{\partial H}{\partial \xi}(x^*, \xi^*, u^*)^T \quad x^*(0) = x_0 \tag{19a}$$

$$\dot{\xi}^* = -\frac{\partial H}{\partial x}(x^*, \xi^*, u^*)^T \tag{19b}$$

$$\xi^*(T) = -\frac{\partial S}{\partial x}(x^*(T))^T \tag{19c}$$

$$\frac{\partial H}{\partial u}(x^*(t), \xi^*(t), u^*(t)) = 0 \tag{19d}$$

Proof In analogy to the Lagrange multiplier method for optimization of a function under a static constraint we define

$$K(t,q,\dot{q}) = \xi^T(f(x,u) - \dot{x}) - L(x,u). \tag{20}$$

Here $q = (x, \xi, u)$ and $\xi : \mathbb{R} \to \mathbb{R}^n$ plays the role of Lagrange multiplier. The variational problem that we want to consider is the minimization of

$$\tilde{J}(q) = S(q(T)) - \int_0^T K(t, q(t), \dot{q}(t))dt, \tag{21}$$

under the constraint $x(0) = x_0$ and subsequently under the additional constraint $x(T) = x_T$. It will turn out that solutions of these problems automatically satisfy the dynamic constraint $\frac{d}{dt}x = f(x,u)$.
Define the Hamiltonian

$$H(x, \xi, u) = \xi^T f(x,u) - L(x,u) \tag{22}$$

The equations (3,4) for the minimization of $\tilde{J}$ are:

$$\frac{\partial K}{\partial x}(t, q(t), \dot{q}(t)) - \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}}(t, q(t), \dot{q}(t))\right) = 0 \tag{23a}$$

$$\frac{\partial K}{\partial \xi}(t, q(t), \dot{q}(t)) - \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\xi}}(t, q(t), \dot{q}(t))\right) = 0 \tag{23b}$$

$$\frac{\partial K}{\partial u}(t, q(t), \dot{q}(t)) - \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{u}}(t, q(t), \dot{q}(t))\right) = 0 \tag{23c}$$

$$\frac{dS}{dx}(q(T)) - \frac{\partial K}{\partial \dot{x}}(T, q^*(T), \dot{q}^*(T)) = 0 \tag{24a}$$

$$\frac{dS}{d\xi}(q(T)) - \frac{\partial K}{\partial \dot{\xi}}(T, q^*(T), \dot{q}^*(T)) = 0 \tag{24b}$$

$$\frac{dS}{du}(q(T)) - \frac{\partial K}{\partial \dot{u}}(T, q^*(T), \dot{q}^*(T)) = 0 \tag{24c}$$

$$\tag{24d}$$

Expanding (23,24) yields:

$$\xi(t)^T \frac{\partial f}{\partial x}((x(t), u(t)) - \frac{\partial L}{\partial x}((x(t), u(t)) + \frac{d}{dt}\xi(t)^T = 0 \tag{25a}$$

$$\dot{x}(t) - f(x(t), u(t)) = 0 \tag{25b}$$

$$\xi(t)^T \frac{\partial f}{\partial u}(x(t), u(t)) - \frac{\partial L}{\partial u}(x(t), u(t)) = 0 \tag{25c}$$

$$\frac{dS}{dx}(x(T)) + \xi(T)^T = 0 \tag{26}$$

By direct inspection it follows that (25a) implies (19b), (25c) is just (4.74), and (26) is the end condition (19c). □

If instead of a penalty on the final state, sometimes referred to as a *soft* constraint, there is a hard constraint of the form $x(T) = x_T$, then the necessary conditions reduce to (23) so that the terminal condition (26) no longer holds. In fact it is replaced by the terminal condition on the state x. The details are left as an exercise.

3 Theorem (Maximum Principle)
Suppose $u^*(\cdot) \in \mathcal{U}$is a solution of the OCP. Then there exists a pair $(x^*(t), \xi^*(t))$ defined on $[0, T]$ such that

$$\dot{x}^* = \frac{\partial H}{\partial \xi}(x^*, \xi^*, u^*)^T \quad x^*(0) = x_0 \tag{27a}$$

$$\dot{\xi}^* = -\frac{\partial H}{\partial x}(x^*, \xi^*, u^*)^T \quad \xi^*(T) = -\frac{\partial S}{\partial x}(x^*(T))^T \tag{27b}$$

$$H(x^*(t), \xi^*(t), u^*(t)) \geq H(x^*(t), \xi^*(t), v), \quad \forall t \in [0, T], \quad \forall v \in U \tag{28}$$

Proof [Sketch only] For notational convenience we present the case for which all variables are scalar. Let (x, ξ, u) be an optimal trajectory where ξ satisfies (27b). Let δu be an admissible input function and denote by $x + \delta x$ the solution of $\dot{x} = f(x, u + \delta u)$, $x(0) = x_0$. We calculate a first order approximation of the increased cost due to δu. First notice that

$$\frac{d}{dt}\delta x(t) = f(x(t)+\delta x(t), u(t)+\delta u(t)) - f(x(t), u(t)) \approx \frac{\partial f}{\partial x}(x(t), u(t))\delta x(t) + \frac{\partial f}{\partial u}(x(t), u(t))\delta u(t) \tag{29}$$

$$\begin{aligned}
0 \leq & J(x_0, u + \delta u) - J(x_0, u) = S(x(T) + \delta x(T)) - S(x(T)) \\
& + \int_0^T L(x(t) + \delta x(t), u(t) + \delta u(t)) - L(x(t), u(t))dt \\
& = S(x(T) + \delta x(T)) - S(x(T)) \\
& + \int_0^T \xi(t)(f(x(t) + \delta x(t), u(t) + \delta u(t)) - f(x(t), u(t))) \\
& - (H(x(t) + \delta x(t), \xi(t), u(t) + \delta u(t)) - H(x(t), \xi(t), u(t)))dt \\
& \approx \frac{dS}{dx}(x(T))\delta x(T) + \int_0^T \xi(t)(\frac{\partial f}{\partial x}(x(t), u(t))\delta x(t) + \frac{\partial f}{\partial u}(x(t), u(t))\delta u(t)) \\
& - (\frac{\partial H}{\partial x}(x(t), \xi(t), u(t))\delta x(t) + \frac{\partial H}{\partial u}(x(t), \xi(t), u(t))\delta u(t))dt \\
& \approx \frac{dS}{dx}(x(T))\delta x(T) + \int_0^T \xi(t)\dot{\delta x}(t) + \dot{\xi}(t)\delta x(t) - \frac{\partial H}{\partial u}(x(t), \xi(t), u(t))\delta u(t))dt \\
& \approx \int_0^T H(x(t), \xi(t), u(t)) - H(x(t), \xi(t), u(t) + \delta u(t))dt
\end{aligned} \tag{30}$$

The last step follows from

$$\frac{dS}{dx}(x(T))\delta x(T) + \int_0^T \xi(t)\dot{\delta x}(t) + \dot{\xi}(t)\delta x(t) = \frac{dS}{dx}(x(T))\delta x(T) + \xi(t)\delta x(t)\Big|_0^T = 0, \tag{31}$$

where we have used the end condition (26) and the fact that $\delta x(0) = 0$.
Since δu is arbitrary it follows, as shown in Lemma 4, that

$$H(x(t), \xi(t), u(t)) - H(x(t), \xi(t), u(t) + \delta u(t)) \geq 0, \tag{32}$$

so that $u(t)$ indeed maximizes $H(x(t), \xi(t), u)$. □

4 Lemma
Let $F, u : \mathbb{R} \to \mathbb{R}$ be continuous functions such that for all continuous $\delta u : \mathbb{R} \to \mathbb{R}$

$$\int_0^T F(u(t)) - F(u(t) + \delta u(t))dt \geq 0, \tag{33}$$

then for all $v \in \mathbb{R}$ and all $t \in \mathbb{R}$ there holds:

$$F(u(t)) \geq F(u(t) + v) \tag{34}$$

Proof Assume the contrary, then there exist a $\bar{t}$ and a $v \in \mathbb{R}$ such that

$$F(u(\bar{t})) < F(u(\bar{t}) + v) \tag{35}$$

Because of continuity there exists $\epsilon > 0$ such that for all t with $|t - \bar{t}| \leq \epsilon$:

$$F(u(t)) < F(u(t) + v) \tag{36}$$

Define the function $\tilde{\delta u}$ as follows:

$$\tilde{\delta u}(t) = v \quad |t - \bar{t}| \leq \epsilon \tag{37}$$
$$= 0 \quad \text{elsewhere} \tag{38}$$

Then

$$\int_0^T F(u(t)) - F(u(t) + \tilde{\delta u}(t))dt < 0. \tag{39}$$

Of course, $\tilde{\delta u}$ is not continuous. This, however, is not a problem. For, $\tilde{\delta u}$ can be approximated arbitrarily good by continuous functions so that there also exists a *continuous* function δu for which (33) is not true. This is a contradiction and the statement follows. □