Exam Optimal Control of Economic Systems, code 156066

Please provide clear motivation for all your answers and indicate which theorems you are using. Do not spend too much time on a single item. If you are not able to solve part(s) of a problem, then move on and use those parts as if you have already solved them.

1. A curve needs be drawn in the plane connecting the points $(x, y) = (0, 2)$ and $(2, 0)$. The curve should reach the zero level as fast as possible, on the other hand the steepness should not be too big. This problem can be modelled as follows: Minimize

$$\int_0^2 y(x) + \dot{y}(x)^2 dx \quad y(0) = 2 \quad y(2) = 0 \tag{1}$$

   (a) Find a function $y(x)$ that satisfies the necessary conditions for optimality.

   (b) Same problem except that now $y(2)$ is free.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x_1, x_2) = (-\sin(x_1) + \sin(x_2), -\sin(x_2)), \tag{2}$$

   and consider the system

$$\dot{x}(t) = f(x(t)) \tag{3}$$

   (a) Calculate all *real* equilibrium points.

   (b) Calculate $\frac{df}{dx}(\bar{x}_1, \bar{x}_2)$ for all equilibrium points $(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2$.

   (c) Investigate the stability of $(0, 0)$.

   (d) Investigate the stability of $(\pi, 0)$.

   (e) Let $A = \frac{df}{dx}(0, 0)$. Solve the Lyapunov equation $A^T P + PA = -4I$.

   (f) Provide a Lyapunov function that proves the asymptotic stability of $(0, 0)$.

   (g) Can $(0, 0)$ be *globally* asymptotically stable?

3. A new light railway needs to be constructed between Enschede and Hengelo. To save material and to minimize traveling time the track is required to be of minimal length. Of course, if Holland, or Twente for that matter, were flat then the solution would be to draw a straight line. However, as everybody knows there is a huge obstacle, formed in the last ice age, between the two cities: the impressive Driener Mountains.

   We model the problem of constructing the railway of minimal length as follows. In the x-y plane we design a curve that we view as the graph of a function: $(x, y(x))$. The Driener mountains, due to erosion, admit a very crisp and clear description: $z = 1 - x^2$. Enschede is situated in $(x, y, z) = (-1, -1, 0)$ and the coordinates of Hengelo are $(1, 1, 0)$. For a given choice of $y(\cdot)$, the curve that the track follows is parameterized as $(x, y(x), 1 - x^2)$. See Figure 1.

   The length of the curve is given by:

$$\int_{-1}^1 \sqrt{1 + \dot{y}(x)^2 + 4x^2}\, dx \tag{4}$$

   (a) Show that the optimal path satisfies:

$$-4x\dot{y}(x) + \ddot{y}(x) + 4x^2\ddot{y}(x) = 0 \quad y(-1) = -1 \quad y(1) = 1 \tag{5}$$
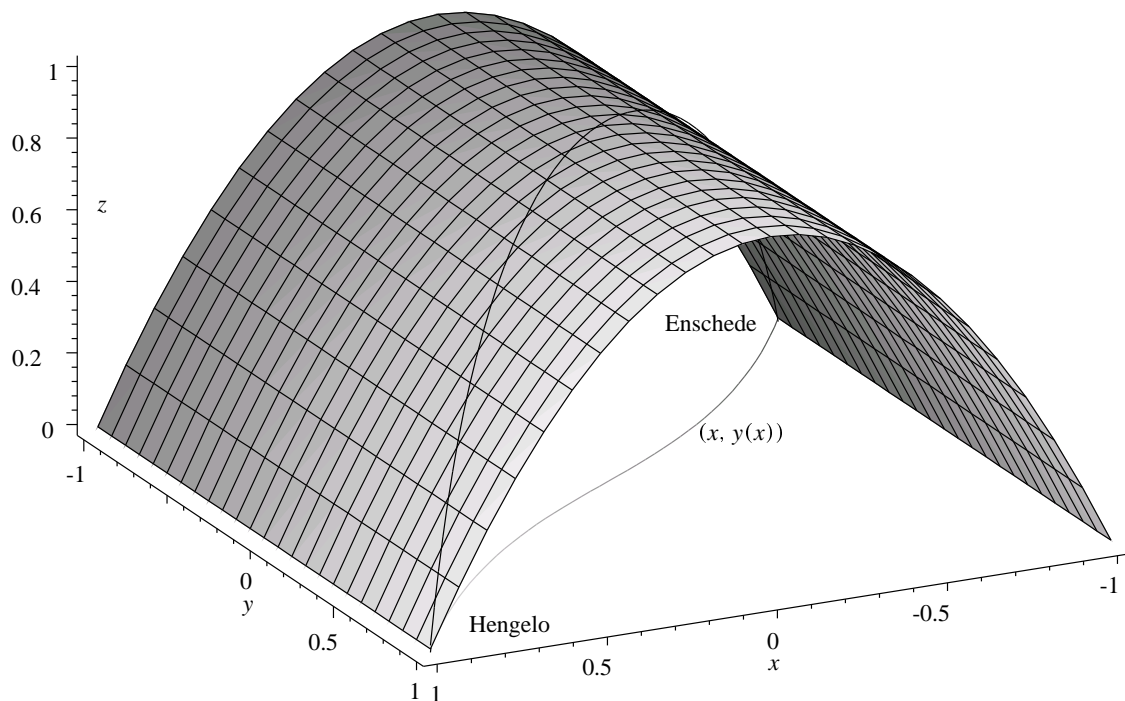
Figure 1: Light railway from Enschede to Hengelo, crossing the rugged Driener Mountains

(b) Substitute $v = \dot{y}$ in (5) and prove (by separation of variables or by inspection) that

$$v(x) = c_1\sqrt{1 + 4x^2} \quad \text{some } c_1 \in \mathbb{R} \tag{6}$$

The optimal path is therefore given by

$$y(x) = c_2 + c_1 \int_{-1}^{x} \sqrt{1 + 4s^2}\, ds \tag{7}$$

4. Consider the system and cost criterion:

$$\frac{d}{dt}x = u \quad J(x_0, u) = \int_0^1 2x(t)^2 + 2x(t)u(t) + u(t)^2 dt \tag{8}$$

   (a) Give the Hamiltonian, determine the optimal control, and derive the differential equations for the co-state.
   (b) Solve the differential equations for the state and co-state.
   (c) Assume that the value function for this problem is of the form $p(t)x^2$. Derive a differential equation for the function p.
   (d) Use the relation between the value function and the co-state to solve the differential equation for p.

Grading:

1		2							3		4			
a	b	a	b	c	d	e	f	g	a	b	a	b	c	d
10	10	4	3	3	3	5	3	2	12	6	10	7	6	6

Grade: $1 + \dfrac{\text{points}}{10}$

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