Asymptotic Methods for Differential Equations

Lecture Notes

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This set of notes is by no means meant as a stand-alone text from which one can hope to ‘learn asymptotics’. It is, instead, a summary (and critical discussion) of the material presented in class. For a proper read, please consult our excellent coursebook (M. Holmes’s ‘Introduction to Perturbation Methods,’ Springer, 1995). Note, though, that an integral part of the course to which these notes correspond is gleaning information from the relevant literature. The transition from coursebook to books and research papers is a subtle (that’s a euphemism for hard) yet necessary step in your development as a researcher. I make an effort below to provide an adequate (yet by no means comprehensive!) list of references, each of which treats appropriately certain parts of the material presented here. Please do your part and do consult this literature.

My aim in teaching this class is neither to be rigorous nor to present asymptotics in a mathematically consistent way—books that attempt to do exactly this do exist, and some of them are included in the bibliography. In fact, there are nearly as many flavors to asymptotics as books on asymptotics; the particular flavor I am interested in for this class is the intuitive one, and this reflects in the notes below. This is not to say that rigor will be absent from the class, but, rather, that our point of view will be functional (‘does this work?’). For this and other reasons, expect to hear the course’s motto—‘know your problem!’—more than once.

At any rate, I hope to help you discover the internal logic of (and let you develop more insight in) the subject at hand. A very good first exercise to that effect is to try and fill in as many gaps in the presentation below as possible—look out, in particular, for bracketed statements. Not all of these statements are easy to answer/prove/substantiate, but you will benefit from working on each one of them.

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Lecture no.1

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1.1 A regularly perturbed algebraic equation

[See Holmes Sect. 1.5] for a similar example.] We start by considering a problem which is fully tractable analytically, namely that of finding the solutions to the quadratic equation

\[ f(\lambda; \varepsilon) = \lambda^2 - 2\lambda + \varepsilon = 0. \]  

(1.1)

Here, \( \varepsilon \) is a parameter taking values in an interval \([0, \bar{\varepsilon}]\), for some \( \bar{\varepsilon} > 0 \). The two solutions of this equation are exactly given by the formula

\[ \lambda^\pm = 1 \pm \sqrt{1 - \varepsilon}, \]

see also Fig. [1].

Note, now, the following. If \( \bar{\varepsilon} < 1 \), then we may Taylor-expand the square root in the formula above to obtain the convergent series

\[ \lambda^\pm = 1 \pm \left( 1 - \sum_{n=1}^{\infty} \frac{\prod_{k=1}^{n-1} (2k - 1)}{n! 2^n} \varepsilon^n \right) = 1 \pm \left( 1 - \frac{1}{2} \varepsilon - \frac{1}{8} \varepsilon^2 - \ldots \right) = \left\{ \begin{array}{l} 2 - \frac{1}{2} \varepsilon - \frac{1}{8} \varepsilon^2 - \ldots \\ 1 + \frac{1}{2} \varepsilon + \frac{1}{8} \varepsilon^2 + \ldots \end{array} \right. \]  

(1.2)

[Why \( \bar{\varepsilon} < 1 \)? What goes wrong for \( \bar{\varepsilon} \geq 1 \)]? Our fundamental observation is that we can obtain the coefficients of the series (1.2) by
(a) substituting the series
\[ \lambda = \sum_{n=0}^{\infty} \varepsilon^n \lambda_n = \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_0 + \ldots \] (1.3)

directly into (1.1);
(b) collecting powers of \( \varepsilon \) in the resulting equation;
(c) setting the coefficient of each such power to zero.
Indeed, here is how the three-point program above can be carried out.

(a) First, substitution of (1.3) into (1.1) yields
Indeed, here is how the three-point program above can be carried out.

\[ 0 = \left[ \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_0 + \ldots \right]^2 - 2 \left[ \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_0 + \ldots \right] + \varepsilon \]
\[ = \left[ \lambda_0^2 + 2 \varepsilon \lambda_0 \lambda_1 + \varepsilon^2 \left( \lambda_1^2 + 2 \lambda_0 \lambda_2 \right) + \ldots \right] - 2 \left[ \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_0 + \ldots \right] + \varepsilon. \]

(b) Rearranging terms, we can rewrite this equation as
\[ \lambda_0^2 - 2 \lambda_0 + \varepsilon \left[ 2 \lambda_0 \lambda_1 - 2 \lambda_1 + 1 \right] + \varepsilon^2 \left[ \lambda_1^2 + 2 \lambda_0 \lambda_2 - 2 \lambda_2 \right] + \ldots = 0. \] (1.4)

(c) Setting the coefficient of each power of \( \varepsilon \) to zero, one obtains a countably infinite set of algebraic equations; the first three are
- \( \lambda_0^2 - 2 \lambda_0 = 0 \), whence \( \lambda_0^+ = 2 \text{ or } \lambda_0^- = 0 \);
- \( 2 \lambda_0 \lambda_1 - 2 \lambda_1 + 1 = 0 \), whence \( \lambda_1^+ = -1/2 \text{ or } \lambda_1^- = 1/2 \), respectively;
- \( \lambda_1^2 + 2 \lambda_0 \lambda_2 - 2 \lambda_2 = 0 \), whence \( \lambda_2^+ = -1/8 \text{ or } \lambda_2^- = 1/8 \), respectively.

Discussion. Plainly, this outcome matches (1.2) up to and including \( \varepsilon^2 \)-terms. The reason underpinning this agreement is not hard to fathom, once one knows that (1.3) is absolutely convergent. Indeed, squaring a series and subsequently rearranging its terms (as we did) are both valid under this condition; the same goes for summing up series. Similarly, (1.4) is true if and only if the coefficient of each power of \( \varepsilon \) is identically zero (essentially by the uniqueness property of Taylor expansions). Thus, our work above is rigorously justified, provided that we know that (1.3) is an absolutely convergent series. To prove absolute convergence, knowledge of (1.1) is unnecessary (although it does not hurt at all, either...): the same information can be deduced by the series’ coefficients, assuming that an appropriate formula for the \( n \)th (general) coefficient has been successfully determined. [How can one prove absolute convergence of a series of which one has the general term?] We will return to this fine point later on, when we discuss asymptotic expansions.

There are certain fine points related to our discussion above. First, one cannot but observe that, for \( \varepsilon > 1 \), (1.1) yields two complex solutions with non-trivial imaginary parts. This seems incompatible with the fact that (1.3) is a real series, until one notices that this series becomes divergent for such values of \( \varepsilon \). That the series is divergent, though, can only be deduced if one can produce a formula for the general coefficient of the series. In this particular example, this is possible; this is not always the case, though. Even worse, in cases where it is possible, one might find that the series diverges for all nonzero values of \( \varepsilon \)—a classic example of such a series is the Stieltjes series \( \sum_{n \geq 0} (-1)^n n! x^n \) (see [Bender & Orszag, Sect. 3.8, Example 3]). We will see further down what such divergent series have to offer.

1.2 A singularly perturbed algebraic equation

[See [Holmes, Sect. 1.5] for a similar example.] Consider, now, the problem
\[ f(\lambda; \varepsilon) = -\varepsilon \lambda^4 + \lambda^2 - 2 \lambda + \varepsilon = 0, \] (1.5)
with \( \varepsilon \) as before: a “small” parameter. Although this equation can also be solved explicitly, we shall not bother with the analytic formula. We will, instead, determine series expansions for the roots to this quartic.
Figure 1: Graphs of the functions $f(\cdot;0)$ (in black) and $f(\cdot;\varepsilon)$ (in red). Note that the latter is a mere vertical shift by $\varepsilon$ of the former.

Equation (1.5) seems to be a perturbation of (1.1), since the additional term, $\varepsilon \lambda^4$, is multiplied by the small parameter $\varepsilon$. Hence, a (hasty...) first guess would be that the solutions of (1.5) are close to the solutions (1.2) of (1.1). (The more careful reader, on the other hand, will not fail to notice that (1.5) must have not two but four roots.) Substituting (1.3) into (1.5), rearranging, and collecting terms, we find

$$[\lambda_0^2 - 2\lambda_0] + \varepsilon [-\lambda_0^4 + 2\lambda_0\lambda_1 - 2\lambda_1 + 1] + \ldots = 0,$$

whence we obtain

$$\lambda_0^+ = 2 + \frac{15}{2}\varepsilon + \ldots \quad \text{and} \quad \lambda_0^- = \frac{1}{2}\varepsilon + \ldots \quad (1.6)$$

Indeed, these two roots seem to be $\varepsilon$—close to those given in (1.2)—no wonder, as both (1.1) and (1.5) read $\lambda_0^2 - 2\lambda_0 = 0$, for $\varepsilon = 0$. In fact, it can be easily proven that there are two roots of (1.5)—say, $\lambda^+(\varepsilon)$ and $\lambda^-(\varepsilon)$—which satisfy

$$\lim_{\varepsilon \downarrow 0} |\lambda^+(\varepsilon) - \lambda_0^+| = 0.$$

Even more strongly, it can be shown that there exist $\varepsilon_0 > 0$ and constants $C^+ > 0$ such that

$$|\lambda^+(\varepsilon) - \lambda_0^+| < C^\varepsilon, \quad \text{for all} \ 0 \leq \varepsilon \leq \varepsilon_0.$$

In other words, the distance between $\lambda^+(\varepsilon)$ and $\lambda_0^+$ scales at least as fast as $\varepsilon$. (In fact, it scales exactly as fast as $\varepsilon$—that is, $C^\varepsilon$ cannot be made arbitrarily small by lowering the value of $\varepsilon_0$, and hence $\lambda^\varepsilon$ converge to their unperturbed values linearly, as $\varepsilon \downarrow 0$.) [Try to prove all of this.]

What about the remaining two roots, which neither were captured by the expansion (1.3) nor are present for $\varepsilon = 0$? To understand our problem at an intuitive level, we return to the idea that $\tilde{f}(\cdot;\varepsilon)$ is a perturbation of $f(\cdot;\varepsilon)$. This is certainly true for every fixed value of $\lambda$, in the sense that

$$\lim_{\varepsilon \downarrow 0} \tilde{f}(\lambda;\varepsilon) = \lambda^2 - 2\lambda = \lim_{\varepsilon \downarrow 0} f(\lambda;\varepsilon), \quad \text{for every} \ \lambda.$$

Employing the definition of the limit, we obtain that, for every $c > 0$, there exists $\varepsilon_0 > 0$ such that

$$|\tilde{f}(\lambda;\varepsilon) - f(\lambda;\varepsilon)| < c, \quad \text{for all} \ 0 \leq \varepsilon \leq \varepsilon_0.$$

(Using logical quantifiers, we may rewrite this in the form

$$\forall c > 0 \forall \lambda \in \mathbb{R} \exists \varepsilon_0 > 0 \forall 0 \leq \varepsilon \leq \varepsilon_0 \left[|\tilde{f}(\lambda;\varepsilon) - f(\lambda;\varepsilon)| < c\right]; \quad (1.7)$$
we will favor the use of quantifiers in the next section, too, so it is suggested you familiarize yourselves with them.) As is always the case when a statement involves the universal ( ∀ ) and the existential ( ∃ ) quantifiers, one has to be careful with interchanging them; in particular, ∀ ∀ ∃ but not vice versa. Here, in particular, it is not true that

\[ \forall \epsilon > 0 \exists \lambda > 0 \forall \lambda \in \mathbb{R} \forall 0 \leq \epsilon \leq \epsilon_0 \left[ |f(\lambda; \epsilon) - f(\lambda; \epsilon)| \right] < c, \]

which would imply that \( \tilde{f}(\cdot; \epsilon) \) and \( f(\cdot; \epsilon) \) are uniformly close over \( \mathbb{R} \). And indeed, the two functions do not remain uniformly close over the entire real line, as the calculation

\[ \lim_{|\lambda| \to \infty} f(\lambda; \epsilon) = -\infty \neq \infty = \lim_{|\lambda| \to \infty} f(\lambda; \epsilon), \quad \text{for all } \epsilon > 0, \]

plainly shows. In other words, no matter how small (but positive) \( \epsilon \) is, there is always a neighborhood of infinity where the functions \( \tilde{f}(\cdot; \epsilon) \) and \( f(\cdot; \epsilon) \) diverge: the former approaches \( -\infty \) (due to the term \( -\epsilon \lambda^4 \), which eventually takes over), while the latter approaches \( \infty \) (due to the term \( \lambda^2 \)). Under these circumstances, there can be no discussion of closeness, see also Fig. 2.

This last remark also offers us the way out of the conundrum. For any fixed value of \( \lambda \) (and for sufficiently small \( \epsilon \)), \( \tilde{f}(\lambda; \epsilon) \) and \( f(\lambda; \epsilon) \) are close to each other; for all \( \lambda \not\in [0, 3] \) and all \( \epsilon \in [0, \bar{\epsilon}] \) (for some \( \bar{\epsilon} > 0 \)), then, \( \tilde{f}(\lambda; \epsilon) > 0 \). Since \( \tilde{f}(\cdot; \epsilon) \) tends to \( -\infty \) in a continuous manner as \( |\lambda| \) increases, it needs to cross zero on its way there; this yields a root smaller than \( \lambda = 0 \) and a root larger than \( \lambda = 3 \). As argued above, the uniform closeness of \( \tilde{f}(\lambda; \epsilon) \) and \( f(\lambda; \epsilon) \) over any bounded set \( I \) implies that these roots cannot be contained in any such set as \( \epsilon \downarrow 0 \). Hence, they must become arbitrarily large (i.e., unbounded) as \( \epsilon \downarrow 0 \). To capture this effect, we rescale \( \lambda \) via \( \lambda = \epsilon^{-a} \Lambda \), for some yet undetermined \( a > 0 \); the exponent \( a \) will be determined by demanding that the values of \( \Lambda \) corresponding to these unbounded roots remain bounded away from both infinity and zero as \( \epsilon \downarrow 0 \). In other words, we assume that these unbounded roots tend to infinity algebraically (as \( \epsilon^{-a} \), for some positive power \( a \)). Under this rescaling, (1.5) becomes

\[ -\epsilon^{1-4a} \Lambda^4 + \epsilon^{-2a} \Lambda^2 - 2\epsilon^{-a} \Lambda + \epsilon = 0. \]

Now, the last term \( (\epsilon) \) limits to zero as \( \epsilon \downarrow 0 \), whereas the next-to-last term \( (2\epsilon^{-a} \Lambda) \) blows up (recall that \( \Lambda \) was assumed bounded away from zero). Hence, the former is a mere perturbation to the latter. Similarly, the second term \( (\epsilon^{-2a} \Lambda^2) \) blows up much faster than \( 2\epsilon^{-a} \Lambda \), as \( \epsilon^{-2a} \) tends to infinity much faster than \( \epsilon^{-a} \) as

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1 The reader should try to show that, nevertheless, \( \tilde{f}(\cdot; \epsilon) \) and \( f(\cdot; \epsilon) \) are uniformly close over any bounded subset \( I \) of \( \mathbb{R} \)—that is, replacing \( \mathbb{R} \) by \( I \) in (1.7) allows us to exchange the two quantifiers. Note that \( \epsilon_0 \) has to be taken smaller as the diameter of \( I \) grows, with \( \epsilon_0 \) limiting to zero as the diameter approaches infinity. This last property often acts as the 'signature' of a problem where uniformity cannot be established.
\( \varepsilon \downarrow 0 \); hence, the next-to-last term is also perturbative to the second term. To obtain bounded, non-zero values of \( \Lambda \), then, one must balance the first and second terms. Since these two terms must balance each other for all \( \varepsilon \in [0, \bar{\varepsilon}] \), it follows that \( 1 - 4a = -2a \) or, equivalently, that \( a = 1/2 \). Under this condition, we may recast (1.9) in the form

\[
-\Lambda^4 + \Lambda^2 - 2\sqrt{\varepsilon}\Lambda + \varepsilon^2 = 0,
\]

which is a regularly perturbed problem like the one treated in the previous section. Postulating for \( \Lambda \) a power series of the form

\[
\Lambda = \Lambda_0 + \sqrt{\varepsilon}\Lambda_1 + \varepsilon\Lambda_2 + \ldots
\]

(note that the “natural” small parameter in (1.10) is \( \sqrt{\varepsilon} \) and not \( \varepsilon \)) and working as in that section, we can produce the expansion

\[
\Lambda = \pm 1 - \sqrt{\varepsilon} + \ldots.
\]

Passing back to the original variable \( \lambda \), then, we find

\[
\lambda = \pm \frac{1}{\sqrt{\varepsilon}} - 1 + \ldots.
\]

For completeness, note that (1.10) also yields two roots which are zero to leading order; the reader can show that these correspond to the two bounded roots reported in (1.6).

**Remark.** The leading order result \( \lambda = \pm \varepsilon^{-1/2} - 1 + \ldots \), which suggests the proper rescaling for \( \lambda \), can be also obtained by the following considerations. First, there exist no extra roots which remain bounded as \( \varepsilon \downarrow 0 \) for the reasons outlined above. Second, for “large” values of \( \lambda \), the dominant term among the second-through-fourth ones in (1.3) is the second (quadratic) one. For a root to exist, \( \lambda \) needs to have such a value that this term (which is large and positive) is balanced by the first (quartic) term; for smaller values of lambda, the quadratic term is dominant and \( f(\lambda; \varepsilon) > 0 \); for larger values of lambda, the dominant term is the quartic and \( f(\lambda; \varepsilon) < 0 \). Thus, \( -\varepsilon\lambda^4 + \lambda^2 \approx 0 \), whence \( \lambda \approx \pm \varepsilon^{-1/2} \).

### 1.3 Order symbols

In this and the next section, we will (briefly) formalize our discussion of approximate solutions and asymptotic expansions. The reader is encouraged to consult any book in the bibliography for additional information on the subject. In what follows, \( D \subset \mathbb{R}^M \) (for some \( M \in \mathbb{N} \)), \( I = (0, \bar{\varepsilon}) \) (for some \( \bar{\varepsilon} > 0 \)), \( u : D \times I \to \mathbb{R} \), and \( v : D \times I \to \mathbb{R} \). First, we introduce the order symbol \( o(\cdot) \) (called ‘little \( \mathcal{O} \)’).

**Definition.** We write

\[
u(\cdot; \varepsilon) = o(v(\cdot; \varepsilon)) \ (\varepsilon \downarrow 0) \text{ over } D \quad \text{if} \quad \forall x \in D \forall k > 0 \exists \varepsilon_0 > 0 \forall \varepsilon < \varepsilon_0 \left[ |u(x; \varepsilon)| < k |v(x; \varepsilon)| \right]. \quad \Box \quad (1.11)
\]

In words, \( u = o(v) \) (‘(\( \varepsilon \downarrow 0 \))’ is often omitted) if, for every \( x \in D \), \( u(x; \cdot) \) can be bounded by any—arbitrarily small—multiple \( k \) of \( v(x; \cdot) \) over some \( (k\text{-dependent, naturally}) \) neighborhood of zero. The relation \( u = o(v) \) is often also written \( u \ll v \), for historical reasons. If the set \( \{ \varepsilon : v(x; \varepsilon) \neq 0 \} \) has, for each \( x \), zero as a limit point, then (1.11) is simply identical to

\[
\forall x \in D \lim_{\varepsilon \downarrow 0} \frac{u(x; \varepsilon)}{v(x; \varepsilon)} = 0.
\]

(In fact, some authors use this relation to define \( o(\cdot) \).) Next, we introduce the order symbol \( \mathcal{O}(\cdot) \) (called ‘big \( \mathcal{O} \)’).

**Definition.** We write

\[
u(\cdot; \varepsilon) = \mathcal{O}(v(\cdot; \varepsilon)) \ (\varepsilon \downarrow 0) \text{ over } D \quad \text{if} \quad \forall x \in D \exists k > 0 \exists \varepsilon_0 > 0 \forall \varepsilon < \varepsilon_0 \left[ |u(x; \varepsilon)| < k |v(x; \varepsilon)| \right]. \quad \Box \quad (1.13)
\]

In words, \( u = \mathcal{O}(v) \) if, for every \( x \in D \), \( u(x; \cdot) \) can be bounded by \( a—\text{large enough}—\)multiple \( k \) of \( v(x; \cdot) \) over some neighborhood of zero. Note that (1.13) does not imply that \( u \) and \( v \) exhibit the same asymptotic
behavior as \( \varepsilon \downarrow 0 \)—for example, \( u(x; \varepsilon) \) could be approaching zero (e.g., \( u(x; \varepsilon) = \varepsilon \)) while \( v(x; \varepsilon) \) could be growing unboundedly (e.g., \( v(x; \varepsilon) = \varepsilon^{-1} \)) as \( \varepsilon \downarrow 0 \). Even if both are approaching zero or growing unboundedly, they could do so at different asymptotic rates—compare \( u(x; \varepsilon) = \varepsilon^2 \) and \( v(x; \varepsilon) = \varepsilon \), in the first case, and \( u(x; \varepsilon) = \varepsilon^{-1} \) and \( v(x; \varepsilon) = \varepsilon^{-2} \) in the second one. To alleviate this, we introduce the stronger symbol \( O \) via

\[
\forall x \in D \left( [u(x; \varepsilon) = O(v(x; \varepsilon)) \ (\varepsilon \downarrow 0)] \land \left( u(x; \varepsilon) \neq o(v(x; \varepsilon)) \ (\varepsilon \downarrow 0)\right) \right). \tag{1.14}
\]

(The reader should nevertheless note that, unfortunately, \( O \) is often used in the literature where \( O \) should have been used…)

**Example.** The following relations can be easily shown to be true:

- For all \( a < b, \varepsilon^b = o(\varepsilon^a) \ (\varepsilon \downarrow 0) \).
- For all \( a \in \mathbb{R}, (\log \varepsilon)^{-1} = o(\varepsilon^a) \ (\varepsilon \downarrow 0) \); additionally, \( \log \varepsilon = o(\varepsilon^a) \ (\varepsilon \to \infty) \) (the logarithm tends to infinity slower than any algebraic function).
- For all \( a \in \mathbb{R} \) and \( c > 0, e^{-c/\varepsilon} = o(\varepsilon^a) \ (\varepsilon \downarrow 0) \) (exponentially small terms are asymptotically smaller than any algebraic term).
- \( \varepsilon \sin (\varepsilon^{-1}) = O(\varepsilon) \ (\varepsilon \downarrow 0) \), while the converse is not true.

As is always the case when dealing with convergence of functions, definitions (1.11) and (1.13) raise the issue of uniformity. Both of these definitions quantify the notion or relative asymptotic magnitude pointwise: for every value of \( x \in D, u(x; \varepsilon) \) can be bounded by any \( (\log(\cdot))/a \) (for \( O(\cdot) \)) multiple of \( v(x; \varepsilon) \) over an interval \((0, \varepsilon_0)\). The interval length \( \varepsilon_0 \) in these definitions is \( x- \) dependent; \( \varepsilon_0 = \varepsilon_0(x) \); in general, no \( \varepsilon_0 \) exists which applies to all \( x \)-equivalently, \( \inf_{\varepsilon \in D} \varepsilon_0(x) = 0 \). (In definition (1.13), similar concerns apply to the \( x- \) dependent value of \( k \): in general, \( \sup_{\xi \in D} k(x) = \infty \).) If such a uniformly valid value of \( \varepsilon_0 \) (and of \( k \), for \( O(\cdot) \) can be found, then we talk of **uniform asymptotic estimates**:

\[
u(\cdot; \varepsilon) = O(v(\cdot; \varepsilon)) \ (\varepsilon \downarrow 0) \text{ uniformly over } D \text{ if } \forall k > 0 \exists \varepsilon_0 > 0 \forall 0 < \varepsilon < \varepsilon_0 \forall x \in D \left[ |u(x; \varepsilon)| < k |v(x; \varepsilon)| \right], \tag{1.15}
\]

and

\[
u(\cdot; \varepsilon) = O(v(\cdot; \varepsilon)) \ (\varepsilon \downarrow 0) \text{ uniformly over } D \text{ if } \exists k > 0 \exists \varepsilon_0 > 0 \forall 0 < \varepsilon < \varepsilon_0 \forall x \in D \left[ |u(x; \varepsilon)| < k |v(x; \varepsilon)| \right]. \tag{1.16}
\]

We will see below that uniformity often characterizes regularly perturbed problems and non-uniformity singularly perturbed ones.

**Example.** Consider the functions

\[
u_1(x; \varepsilon) = \varepsilon x + e^{-x/\varepsilon}, \quad u_2(x; \varepsilon) = x + e^{-x/\varepsilon}, \quad \text{and } v(x; \varepsilon) = x, \quad \text{with } D = (0,1) \text{ and } \varepsilon = 1.
\]

- First, \( u_1(\cdot; \varepsilon) = o(v(\cdot; \varepsilon)) \) over \( D \), as

\[
|u_1(x; \varepsilon)| = \varepsilon x + e^{-x/\varepsilon} \leq \left( \varepsilon + x^{-1} e^{-x/\varepsilon} \right) x = \left( \varepsilon + x^{-1} e^{-x/\varepsilon} \right) |v(x; \varepsilon)|.
\]

Hence, we can take \( k(x) = \varepsilon + x^{-1} e^{-x/\varepsilon} \), which, for any \( x \in (0,1) \), can be made arbitrarily small provided that \( \varepsilon \) is small enough. More rigorously, given any \( x \in (0,1) \) and \( k > 0 \), we can use \( \varepsilon_0 = \min(k/2, x/\ln(kx/2)) \) to satisfy (1.11). This is a sharp estimate and \( \varepsilon_0 \downarrow 0 \) as \( x \downarrow 0 \) (because \( x/\ln(kx/2) \downarrow 0 \)). It follows that \( u_1(\cdot; \varepsilon) = o(v(\cdot; \varepsilon)) \) non-uniformly over \( (0,1) \). Intuitively, this is to be expected, as \( v(x; \varepsilon) \to 0 \) and \( u(x; \varepsilon) \to 1 \) for all \( \varepsilon > 0 \), however small; hence, bounding the latter by the former uniformly over \((0,1)\) is not possible.

- Similarly, \( u_2(x; \varepsilon) = O(v(x; \varepsilon)) \) since

\[
|u_2(x; \varepsilon)| = x + e^{-x/\varepsilon} \leq \left( 1 + x^{-1} e^{-x/\varepsilon} \right) x = \left( 1 + x^{-1} e^{-x/\varepsilon} \right) |v(x; \varepsilon)|,
\]

so that (1.13) is satisfied with \( k(x) = 1 + x^{-1} e^{-x/\varepsilon} \) and \( \varepsilon_0 = \varepsilon \). This is a sharp estimate and \( \lim_{x \downarrow 0} k(x) = \infty \), so that the order relation above only holds non-uniformly over \( D \).

- Note, also, that \( u_1(x; \varepsilon) = O(u_2(x; \varepsilon)) \) uniformly over \( D \), since \( |u_1(x; \varepsilon)| = u_1(x; \varepsilon) \leq u_2(x; \varepsilon) = |u_2(x; \varepsilon)| \).
Remark. It is worth mentioning here that the pointwise asymptotic relation \( u(x; \varepsilon) = \mathcal{O}(v(x; \varepsilon)) \) over a compact set \( D \) does not imply that \( u(x; \varepsilon) = \mathcal{O}(v(x; \varepsilon)) \) uniformly over \( D \), as one would expect based on the argument that \( k(x) \) and \( k(x) \) should depend continuously on \( x \), so the former is bounded below and the latter above by positive constants which can be used when employing (1.16).’ See [Holmes Thm. 1.3] for details [and provide a counterexample].

1.4 Asymptotic expansions

Having defined the order symbols \( \mathcal{O}(\cdot) \) and \( o(\cdot) \), we now turn to formalizing the notion of successive approximations. We start with making precise what we mean with (asymptotic) approximation. Here, \( D \) and \( I \) remain defined as in the previous section, while \( f : D \times I \rightarrow R \) and \( \phi : D \times I \rightarrow R \) are functions.

Definition. We write

\[
f(\cdot; \varepsilon) \sim \phi(\cdot; \varepsilon) \ (\varepsilon \downarrow 0) \text{ over } D \quad \text{if} \quad \forall_{x \in D} \left[ f(x; \varepsilon) - \phi(x; \varepsilon) = o(\phi(x; \varepsilon)) \ (\varepsilon \downarrow 0) \right]. \quad (1.17)
\]

In words, \( f \sim \phi \) (‘\( \phi \) is an asymptotic approximation to \( f \)’) if, for every \( x \in D \), the difference \( f(x; \cdot) - \phi(x; \cdot) \) of the two is asymptotically smaller than \( \phi(x; \cdot) \) over some neighborhood of zero. If, in particular, the ratio \( (f - \phi)/\phi \) is well-defined over \( D \times I^{'} \), with \( I^{'} \subset I \) a set having zero as its limit point, then we can employ (1.12) to rewrite (1.17) as

\[
f(\cdot; \varepsilon) \sim \phi(\cdot; \varepsilon) \ (\varepsilon \downarrow 0) \text{ over } D \quad \text{if} \quad \forall_{x \in D} \left[ \lim_{\varepsilon \downarrow 0} \frac{f(x; \varepsilon) - \phi(x; \varepsilon)}{\phi(x; \varepsilon)} = 0 \right]. \quad (1.18)
\]

Note that neither (1.17) nor (1.18) imply that the difference \( f - \phi \) is \( o(1) \) (“small”); they do imply, instead, that this difference becomes (arbitrarily) small compared to \( \phi \) (which asymptotically approximates \( f \)) as \( \varepsilon \downarrow 0 \).

We now proceed with the notion of an asymptotic sequence.

Definition. We call the sequence of functions \( \{\phi_n : I \rightarrow \mathbb{R}\}_{n \geq 0} \) an asymptotic sequence if

\[
\forall_{m \geq 0} \forall_{n \geq m} \left[ \phi_n(\varepsilon) = o(\phi_m(\varepsilon)) \ (\varepsilon \downarrow 0) \right]. \quad (1.19)
\]

Equivalently, \( \{\phi_n\}_{n \geq 0} \) is an asymptotic sequence if

\[
\forall_{m \geq 0} \left[ \phi_{m+1}(\varepsilon) = o(\phi_m(\varepsilon)) \ (\varepsilon \downarrow 0) \right].
\]

An interesting fact related to asymptotic sequences is described by a theorem due to [du Bois-Reymond Verhulst, Sect. 15.1] which states that, for every asymptotic sequence \( \{\phi_n\}_{n \geq 0} \), there exists a function \( \phi : I \rightarrow \mathbb{R} \) satisfying

\[
\forall_{n \geq 0} [\phi(\varepsilon) = o(\phi_n(\varepsilon)) \ (\varepsilon \downarrow 0)].
\]

Note that a similar result is not true for the real numbers, as there is no positive number which is smaller than all members of a positive sequence converging to zero; similarly, there is no positive function which is smaller than all members of a sequence of positive functions converging pointwise to zero.

Remark. Verhulst [Verhulst Sect. 2.5] gives another definition of an asymptotic sequence involving a second sequence, \( \{\delta_n : I \rightarrow \mathbb{R}\}_{n \geq 0} \), of functions which are positive, continuous, monotone and for each of which either \( \lim_{\varepsilon \downarrow 0} \delta_n(\varepsilon) = 0 \) or \( 1/\lim_{\varepsilon \downarrow 0} \delta_n(\varepsilon) \) exists. In these notes, we will mostly only use asymptotic sequences satisfying these conditions, so his alternative definition coincides with (1.19).

Next, we define the notion of asymptotic equality between two functions and with respect to a given asymptotic sequence.
Definition. Let \( f \) and \( \phi \) be as above, \( \{ \phi_n \}_{n \geq 0} \) be an asymptotic sequence, and \( N \in \mathbb{N}^* = \mathbb{N} \cup \{0\} \). Write
\[
f(\cdot; \varepsilon) = \phi(\cdot; \varepsilon) \quad (\varepsilon \downarrow 0) \quad \{ \phi_n \}_{n \geq 0} \quad (N)\] if \( \forall x \in D \left[ f(x; \varepsilon) - \phi(x; \varepsilon) = o(\phi_N(\varepsilon)) \quad (\varepsilon \downarrow 0) \right] \). \( \square \)  
(1.20)

In words, \( f \) and \( \phi \) are asymptotically equal with respect to \( \{ \phi_n \}_{n \geq 0} \) up to and including order \( N \) if their difference is asymptotically smaller than \( \phi_N \) ([Wong Sect. I.3]).

Using this definition, we can now easily introduce the notion of an asymptotic expansion for a given function \( f \) and with respect to a given asymptotic sequence \( \{ \phi_n \}_{n \geq 0} \). In particular, let \( \{ a_n \} \rightarrow \mathbb{R} \}_{n \geq 0} \) be a sequence of functions over \( D \). According to \([1.20]\), then, \( f \) and \( \sum_{0 \leq n \leq N} a_n(x) \phi_n(\varepsilon) \) (for some \( N \in \mathbb{N}^* \)) are asymptotically equal (with respect to \( \{ \phi_n \}_{n \geq 0} \)) up to and including order \( N \) if \( f - \sum_{0 \leq n \leq N} a_n(x) \phi_n(\varepsilon) = o(\phi_N) \). This leads naturally to the idea that \( \sum_{n \geq 0} a_n(x) \phi_n(\varepsilon) \) is the asymptotic expansion of \( f \) if the two are asymptotically equal to all orders.

Definition. Let \( \{ \phi_n \}_{n \geq 0} \) be an asymptotic sequence, \( f : D \times I \rightarrow \mathbb{R} \), and \( \{ a_n : D \rightarrow \mathbb{R} \}_{n \geq 0} \). We write
\[
f(\cdot; \varepsilon) = \sum_{n \geq 0} a_n(\cdot) \phi_n(\varepsilon) \quad (\varepsilon \downarrow 0) \quad D \quad \forall x \in D \forall n \in \mathbb{N}^* \quad \left[ f(x; \varepsilon) - \sum_{n=0}^{N} a_n(x) \phi_n(\varepsilon) = o(\phi_N) \quad (\varepsilon \downarrow 0) \right] . \quad \square \] (1.21)

That is, \( \sum_{n \geq 0} a_n \phi_n \) is the asymptotic expansion of \( f \) if the difference between \( f \) and every truncated sum \( \sum_{0 \leq n \leq N} a_n \phi_n \) is asymptotically smaller than the asymptotically smallest term in that truncated sum.

Note that we talk here of the asymptotic expansion of \( f \). Indeed, given an asymptotic sequence, there is at most one asymptotic expansion for \( f \) with respect to that sequence. The coefficients \( \{ a_n \}_{n \geq 0} \) are calculated iteratively using the formula
\[
a_{N+1}(x) = \lim_{\varepsilon \downarrow 0} \frac{f(x; \varepsilon) - \sum_{0 \leq n \leq N} a_n(x) \phi_n(\varepsilon)}{\phi_{N+1}(\varepsilon)} , \quad \text{for } N \geq 0, \] (1.22)

that is,
\[
a_0(x) = \lim_{\varepsilon \downarrow 0} \frac{f(x; \varepsilon)}{\phi_0(\varepsilon)} , \quad a_1(x) = \lim_{\varepsilon \downarrow 0} \frac{f(x; \varepsilon) - a_0(x) \phi_0(\varepsilon)}{\phi_1(\varepsilon)} , \quad a_2(x) = \lim_{\varepsilon \downarrow 0} \frac{f(x; \varepsilon) - (a_0(x) \phi_0(\varepsilon) + a_1(x) \phi_1(\varepsilon))}{\phi_2(\varepsilon)}, \quad \text{and so on.}
\]

Remark. Some authors (e.g., [Wong]) use, in \([1.21]\), the notation ‘\( \sim \)’ instead of ‘\( = \)’. This use of ‘\( \sim \)’ will be avoided in these notes as, first, it is less common in the literature; and second, it clashes with \((1.17)\). Indeed, the relation ‘\( f(\cdot; \varepsilon) \sim \sum_{n \geq 0} a_n(\cdot) \phi_n(\varepsilon) \quad (\varepsilon \downarrow 0) \) over \( D \)’ introduced in \([1.21]\) can be rewritten as
\[
\forall N \in \mathbb{N}^* \left[ f(\cdot; \varepsilon) - \sum_{n=0}^{N-1} a_n(\cdot) \phi_n(\varepsilon) \sim a_N(\cdot) \phi_N(\varepsilon) \quad (\varepsilon \downarrow 0) \right] \text{ over } D \] (1.23)
[prove this]. In particular, \([1.21]\) does not imply the relation \( f(\cdot; \varepsilon) \sim \sum_{n=0}^{N} a_n(\cdot) \phi_n(\varepsilon) \), except for \( N = 0 \). To see this, note that—as long as \( a_0 \neq 0 \) (which is generically true)—the relation \( f(\cdot; \varepsilon) \sim \sum_{n=0}^{N} a_n(\cdot) \phi_n(\varepsilon) \) translates into
\[
f(\cdot; \varepsilon) - \sum_{n=0}^{N} a_n(\cdot) \phi_n(\varepsilon) = o \left( \sum_{n=0}^{N} a_n(\cdot) \phi_n(\varepsilon) \right) = o(\phi_0(\varepsilon)),
\]
where in the last step we used that \( \phi_n(\varepsilon) = o(\phi_0(\varepsilon)) \), for all \( 1 \leq n \leq N \). This relation matches \((1.23)\) if and only if \( N = 0 \). In that sense, the notion of an asymptotic sequence generalizes the notion of an asymptotic approximation.

\[\footnote{In the same section, Wong treats shortly (but sufficiently critically) the theory of \textit{generalized asymptotic expansions}; the interested reader is referred there for details.}\]
Remark. The reader is strongly encouraged to look up the excellently concise discussion on the differences between convergent and asymptotic sequences in [Bender & Orszag Sect. 3.8]. In short, a convergent series $\sum_{n \geq 0} c_n$ converges without reference to a specific function $f$; in fact, a function $f$ can be defined by this series, $f(x) = \sum_{n \geq 0} c_n(x)$. A characteristic example would be a convergent power series, $\sum_{n \geq 0} a_n x^n$; such series are often used to express solutions to regular ODEs, implicitly-defined analytic functions, et cetera. To the contrary, a series cannot be ‘asymptotic’ without reference to such a specific function. Take, for example, a non-convergent power series $\sum_{n \geq 0} a_n x^n$ as above. As Bender and Orszag show, no matter how $\{a_n\}_{n \geq 0}$ is chosen, a function $f$ can be constructed with $\sum_{n \geq 0} a_n x^n$ as its asymptotic series. Since, then, all power series are ‘asymptotic’ if no reference to a specific function is made, employing the term ‘asymptotic’ without such a reference is nonsensical. An interesting corollary to this is that asymptotics cannot guarantee the existence of a solution to a given problem; this usually has to be ascertained in another manner, before its asymptotic expansion is determined.

1.5 A singularly perturbed transcendental equation

[See [Kevorkian & Cole Sect. 1.2] for a very similar example.] Consider now the problem of solving the transcendental equation

$$g(x, y; \varepsilon) = y - x - \varepsilon \sin(y) + \frac{\varepsilon}{y - 1} = 0, \text{ with } x \in [0, 2], \tag{1.24}$$

in order to express $y$ in terms of $x$. Note that, compared to the problems we treated in Sect. 1.1 and 1.2, our problem contains an additional variable $x$ (acting as a parameter in this problem); further, we have no information on the number of branches $y = y(x)$ [1.24] admits as solutions. Note, though, that this problem enjoys certain properties which make it analytically tractable, see the remark below for details.

For $\varepsilon = 0$, the equation becomes $y - x = 0$ and admits the unique solution (branch) $y = y_0(x) = x$. The Implicit Function Theorem guarantees, then [work out the details], that [1.24] has a solution $y = y(x)$ for all $\varepsilon$ in a neighborhood of zero and for any $x \in (0, 1) \cup (1, 2)$. To derive an asymptotic expansion for the solution $y(x)$ to the perturbed problem, we write

$$y(x) = y_0(x) + \phi_1(\varepsilon) y_1(x) + o(\phi_1(\varepsilon)), \tag{1.25}$$

where the order function $\phi_1$—the next member of the asymptotic sequence $\{\phi_n\}_{n \geq 0}$ starting with $\phi_0 \equiv 1$—is yet to be determined; at this point, our only requirement is that $\phi_1 \ll \phi_0 = 1$. Substituting [1.25] into [1.24], we obtain

$$[x + \phi_1(\varepsilon) y_1(x) + o(\phi_1(\varepsilon))] - x - \varepsilon \sin(x + \phi_1(\varepsilon) y_1(x) + o(\phi_1(\varepsilon))) + \frac{\varepsilon}{(x - 1) + \phi_1(\varepsilon) y_1(x) + o(\phi_1(\varepsilon))} = 0.$$

Taylor-expanding the sinusoidal term and the fraction, we find further

$$0 = [\phi_1(\varepsilon) y_1(x) + o(\phi_1(\varepsilon))] - \varepsilon [\sin(x) + O(\phi_1(\varepsilon))] + \varepsilon \left[ \frac{1}{x - 1} + O(\phi_1(\varepsilon)) \right]$$

$$= \phi_1(\varepsilon) y_1(x) + \varepsilon \left[ \frac{1}{x - 1} - \sin(x) \right] + o(\phi_1(\varepsilon)),$$

where we have also absorbed all $\varepsilon O(\phi_1(\varepsilon)) = O(\varepsilon \phi_1(\varepsilon))$ terms into $o(\phi_1(\varepsilon))$. The only possible matching is $\phi(\varepsilon) = O(\varepsilon)$, and for simplicity we take $\phi(\varepsilon) = \varepsilon$; then, $y_1(x) = \sin(x) - 1/(x - 1)$, and hence

$$y(x) = x + \varepsilon \left[ \sin(x) - \frac{1}{x - 1} \right] + o(\phi_1(\varepsilon)). \tag{1.26}$$

This process can be repeated as many times as needed to determine higher order terms in the asymptotic expansion (and the asymptotic sequence $\{\phi_n\}_{n \geq 0}$ itself).

It should already be apparent that the two-term expansion [1.26] is neither defined over $[0, 2]$—$x_\ast = 1$ proves problematic—nor has any chance of being uniformly valid in any set with $x_\ast = 1$ as a limit point—e.g., in $(0, 1)$. Indeed, the term $1/(x - 1)$ blows up near $x_\ast = 1$ and the asymptotic expansion stops being well-ordered, as $\varepsilon/(x - 1)$ can easily become comparable to $x \approx 1$ or even much larger. In retrospect, the function
of two variables $g(x, y; \varepsilon)$ can also not be said (for any $\varepsilon > 0$) to be uniformly close to $y - x$ due to this same term. Recalling Sect. [1.2]—where such a non-uniformity gave rise to two additional roots—we decide to zoom in near the root of all evil ($x_*=1$) and examine the situation closer. To achieve this, we rescale the independent variable $x$ via

$$x = x_* + \chi(\varepsilon) X = 1 + \chi(\varepsilon) X,$$

for some $o(1)$ order function $\chi(\varepsilon)$. (Under this rescaling, $x = 1$ is mapped to $X = 0$, and any $O(1)$ change in $X$ yields $O(\chi(\varepsilon)) = o(1)$ changes in $x$—whence the zooming in.) Since the Implicit Function Theorem fails at $(x_*, y_*) = (1, 1)$, we also rescale $y$ around $y_*=1$,

$$y = y_* + \psi(\varepsilon) Y = 1 + \psi(\varepsilon) Y,$$

for some $O(1)$ order function $\psi(\varepsilon)$.

We emphasize here that we will focus on $O(1)$ values of both $X$ and $Y$ that remain bounded away from zero.

Plugging into (1.24), we find

$$\psi(\varepsilon) Y - \chi(\varepsilon) X - \varepsilon \sin(1 + \psi(\varepsilon) Y) + (\psi(\varepsilon))^{-1} \varepsilon Y^{-1} = 0.$$

We will first investigate the leading order behavior, $Y(x) = Y_0(x) + o(1)$. To this effect, we may omit the sinusoidal term as it is asymptotically smaller than the last term by virtue of $\varepsilon \ll \varepsilon \psi(\varepsilon))^{-1}$ (indeed, $\sin(y)$ in the original equation remains bounded near $y_* = 1$, while $1/(y-1)$ grown unboundedly); the resulting equation is a quadratic in disguise,

$$\psi(\varepsilon) Y_0 - \chi(\varepsilon) X + \varepsilon \psi(\varepsilon))^{-1} Y_0^{-1} = 0. \quad (1.27)$$

We consider the following distinct cases:

- $\chi(\varepsilon) \ll \psi(\varepsilon)$: then, the only possible matching in (1.27) is between the first and third terms, whence $\psi(\varepsilon) = O(\sqrt{\varepsilon})$. Since both are of the same sign, though, we find that there exist no real solutions. Hence, when $o(\sqrt{\varepsilon})$—close to $x_*$, there are no solution branches at a distance asymptotically larger than $\chi(\varepsilon)$ from $y_*$. 

- $\chi(\varepsilon) \gg \psi(\varepsilon)$: then, the only possible matching is between the second and third terms, whence $\chi(\varepsilon) \psi(\varepsilon) = O(\varepsilon)$ and $Y_0(x) = 1/X$—equivalently,

$$y(x) = 1 + \frac{\varepsilon}{x - 1} + o(\varepsilon). \quad (1.28)$$

This solution branch appears, then, at a distance asymptotically larger than $\sqrt{\varepsilon}$ from $x_*$—recall that $\chi(\varepsilon) \psi(\varepsilon) = O(\varepsilon)$ and $\chi(\varepsilon) \gg \psi(\varepsilon)$—and lies asymptotically closer to $y_*$ than $x$ does to $x_*$. Note that, for $x$ close to one, it suffers from the same uniformity problems as the branch identified in (1.26).

- $\chi(\varepsilon) = O_*(\psi(\varepsilon))$: in that case, we can let $\chi(\varepsilon) = \psi(\varepsilon)$, so that (1.27) becomes

$$Y_0 - X + \varepsilon \psi(\varepsilon))^{-2} Y_0^{-1} = 0.$$

There are now three sub-cases to consider:

- the case $\varepsilon \psi^{-2}(\varepsilon) \gg 1$—equivalently, $\psi(\varepsilon) \ll \sqrt{\varepsilon}$—is untenable, as the leading order term is the third one which cannot be matched by the remaining two; hence, there are no solutions $o(\sqrt{\varepsilon})$—close to $x_*$ (this extends our earlier result);

- the case $\varepsilon \psi^{-2}(\varepsilon) \ll 1$—equivalently, $\psi(\varepsilon) \gg \sqrt{\varepsilon}$, on the other hand, yields the solution $Y = X$. This corresponds to the branch $x = y$—recall that $x = 1 + \chi(\varepsilon) X$, $y = 1 + \psi(\varepsilon) Y$, and $\chi(\varepsilon) = \psi(\varepsilon)$—and so we establish that this branch extends to a point which approaches $x_*$, as $\varepsilon \downarrow 0$, no faster than $\sqrt{\varepsilon}$. Additionally, at distances asymptotically larger than $\sqrt{\varepsilon}$ from $x_*$, the only solution branch for which $x - 1$ and $y - 1$ are commensurate is the one we had already identified;

---

3The reader can think of this requirement as a way to make the decompositions of $x$ and $y$ more concrete. In particular, each one of these variables is decomposed into two parts—$X$ and $\chi$ for $x$, $Y$ and $\psi$ for $y$. Naturally, there is no unique way to do this. The constraints $0 \neq X = O(1)$ and $0 \neq Y = O(1)$ ensure that $\chi(x)$ and $\psi(y)$ measure the proximity of $x$ and $y$ to $x_* = 1$ and $y_* = 1$, respectively.
– finally, the case $\psi^{-2}(\varepsilon) = O(1)$—equivalently, $\psi(\varepsilon) = O(\sqrt{\varepsilon})$—yields the quadratic

$$Y_0^2 - X Y_0 + 1 = 0, \quad \text{with solutions} \quad Y_0^\pm = \frac{X}{2} \pm \sqrt{\left(\frac{X}{2}\right)^2 - 1}.$$  

Note that the regime $X \gg 1$ corresponds to the regime $\psi(\varepsilon)X \gg \sqrt{\varepsilon}$, and the solutions above become (upon Taylor-expanding them—recall that $X^{-1} = o(1)$ in this regime)

$$Y_0^\pm = \frac{X}{2} \pm \frac{X}{2} \sqrt{1 - \left(\frac{2}{X}\right)^2} = \begin{cases} X + o(X^{-1}), \\ X^{-1} + o(X^{-3}). \end{cases} \quad (1.29)$$

The first among these corresponds to the branch reported in (1.26), whereas the second of these corresponds to the branch reported in (1.28).

**Discussion.** To summarize the results of our investigation near $(x_*, y_*)$, we note the following (see also Fig. 3): first, the solution branch with asymptotic expansion as in (1.26) extends $\sqrt{\varepsilon}$–close to $x_*$. In fact, it extends all the way to two points $x_- < x_*$ (from the left) and $x_+ > x_*$ (from the right), where $x_+ = 1 \pm 2\sqrt{\varepsilon} + o(\sqrt{\varepsilon})$. (These points correspond to the $X$–values for which the root in (1.29) disappears. The higher order terms in their expansions derive from the higher order sinusoidal term which we omitted in writing (1.27).) The branch remains uniformly $O(\sqrt{\varepsilon})$–close over $[0, x_-] \cup [x_+, 2]$ to the leading order approximation $y = x$, albeit only in a $C^1$–fashion. Why does $C^1$–proximity fail? How can one deduce this from Fig. 3? Additionally, we identified a second solution branch with asymptotic expansion reported in (1.28) and extending up to the same points $x_{\pm}$. This second branch remains uniformly $O(\sqrt{\varepsilon})$–close (also in a $C^1$–fashion) over $[0, x_-] \cup [x_+, 2]$ to the leading order approximation $y = 1$. At $x_{\pm}$, these two solution branches meet and annihilate each other.

Note, further, that all cases we considered above turned out to be subcases of the final one. Indeed, both branches we identified for $x - x_* \gg \sqrt{\varepsilon}$—namely $Y(x) \sim X$ and $Y(x) \sim X^{-1}$—are limiting cases of the two-branched solution reported in (1.29). Kevorkian and Cole [Kevorkian & Cole] call this the principle of least degeneracy, since this final case is the only one in which all three terms play a role (i.e., are of the same asymptotic magnitude). Additionally, they work out the matching of each branch for their example: that is, they show that the formulas for $Y(X)$—which are valid $O_*(\varepsilon)$–close to $x_*$ in our case—match their counterparts $y(x)$ at an $O_*(1)$–distance from $x_*$ over an intermediate lengthscale. We will discuss the role of matching in the context of singularly perturbed ODEs at a later section; until then, we remark that expressing the solutions identified in (1.29) in terms of $x$ and $y$ leads to (1.26) and (1.28), up to and including $O_*(\varepsilon)$–terms.

**Remark.** Note that (1.24) is explicitly solvable for $x$,

$$x(y) = y - \varepsilon \sin(y) + \frac{\varepsilon}{y-1}, \quad (1.30)$$

which facilitates the analysis of (1.24). Further, one can shed more light into the singularly perturbed nature of (1.24) by rewriting it as

$$\tilde{g}(x, y; \varepsilon) = (y - 1)(y - x) + \varepsilon [1 - (y - 1)\sin(y)] = 0, \quad \text{with} \quad x \in [0, 2].$$

In this form, it is directly apparent that the solution $y = y(x)$ has two branches. Note that $\partial_y \tilde{g}(1, 1; 0) = 0$, and hence the Implicit Function Theorem remains inapplicable at $x_*$—as it should, since for every $\varepsilon > 0$ there is a $O(\sqrt{\varepsilon})$ neighborhood of $x_*$ where the equation above has no solutions (recall our work above).
Figure 3: The exact solution to \(1.24\) as given in \(1.30\) (in black), together with the two-term asymptotic expansions of the lower (in red) and upper (in blue) branches, see \(1.26\) and \(1.28\), respectively.

**Homework set no.1**

**01.** Consider the quadratic equation

\[
(1 - \varepsilon)x^2 - 2x + 1 = 0.
\]

This is a regularly perturbed problem, and thus we anticipate all roots to remain bounded as \(\varepsilon \downarrow 0\). Set, then,

\[
x = \sum_{n \geq 0} c_n \varepsilon^n,
\]

substitute into the equation, and work out the equations for \(c_0\), \(c_1\), and \(c_2\). What goes wrong? Explain the reason this approach fails and devise a way to rectify it.

**02.** Consider the transcendental equation

\[
\frac{x}{x - 1} - e^{x - 1/\varepsilon} - \varepsilon = 0, \quad \text{where } x \in \mathbb{R}.
\]

(a) Derive an asymptotic expansion for any bounded roots \(x(\varepsilon)\) this equation might have. Find the coefficient of the \(n^{th}\) ("general") term and show that your asymptotic expansion converges for all \(\varepsilon\) in a neighborhood of zero. Does this asymptotic expansion converge to \(x(\varepsilon)\) for \(\varepsilon \neq 0\)?

(b*) Does the equation above also have unbounded roots? If it does, can you derive asymptotic expansions for them?

**03.** [Holmes, Sect. 1.5, Ex. 9] Let \(A, D\) be real, \(n \times n\) matrices.

(a*) Suppose \(A\) is symmetric and has \(n\) distinct eigenvalues. Find a two-term asymptotic expansion of the eigenvalues of the perturbed matrix \(A + \varepsilon D\), where \(D\) is positive definite. What you are finding is known as a Rayleigh–Schrödinger series for the eigenvalues.

(b) Suppose \(A\) is the identity and \(D\) is symmetric. Find a two-term asymptotic expansion of the eigenvalues of the matrix \(A + \varepsilon D\).

(c) By considering

\[
A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]
show that a $O(\varepsilon)$ perturbation of a matrix need not result in a $O(\varepsilon)$ perturbation of the eigenvalues. This example also demonstrates that a smooth (smooth here means differentiable) perturbation of a matrix need not result in a smooth perturbation of the eigenvalues.

04. Let $D \subset \mathbb{R}^M$ (for some $M \in \mathbb{N}$) and $I = (0, \varepsilon_0)$ (for some $\varepsilon_0 > 0$), and consider functions $f$, $g$, $\phi$, and $\gamma$ all of which are defined on $D \times I$ and real-valued. Assume that $f(x, \varepsilon) = O(\phi(x, \varepsilon))$ and $g(x, \varepsilon) = O(\gamma(x, \varepsilon))$, both as $\varepsilon \downarrow 0$.

(a) Show that $f(x, \varepsilon)g(x, \varepsilon) = O(\phi(x, \varepsilon)\gamma(x, \varepsilon))$, as $\varepsilon \downarrow 0$.

(b) Show that $f(x, \varepsilon) + g(x, \varepsilon) = O(|\phi(x, \varepsilon)| + |\gamma(x, \varepsilon)|)$ but $f(x, \varepsilon) + g(x, \varepsilon) \neq O(\phi(x, \varepsilon) + \gamma(x, \varepsilon))$, in general (both as $\varepsilon \downarrow 0$).

(c∗) Recall our definition of the symbol $O_s$. Verhulst [Verhulst Sect. 2.1] also introduces the symbol $\approx$ with the meaning

$$f \approx g \ (\varepsilon \downarrow 0) \quad \text{if and only if} \quad [f(\varepsilon) = O(g(\varepsilon)) \ (\varepsilon \downarrow 0)] \land [g(\varepsilon) = O(f(\varepsilon)) \ (\varepsilon \downarrow 0)].$$

Are the two relations equivalent? If not, does one of the relations $f = O_s(g)$ and $f \approx g$ (both as $\varepsilon \downarrow 0$) imply the other? If yes, prove it; if not, provide a counterexample.
2.1 A regularly perturbed ODE

We now turn to perturbed ODEs and consider a simple boundary-value problem involving a linear ODE,

\[ y'' - \varepsilon y' - y = 0, \quad \text{with} \quad y(0) = -1, \quad y(1) = 1. \]  

(2.1)

The general solution to the linear ODE is

\[ y(x; \varepsilon) = c_+ e^{\lambda_+ x} + c_- e^{\lambda_- x}, \quad \text{where} \ c_+ \text{ and } c_- \text{ are arbitrary constants.} \]

Here, \( \lambda_\pm = \varepsilon/2 \pm \sqrt{1 + \varepsilon^2/4} \) are the roots of the characteristic polynomial. Employing the boundary conditions, we find

\[ y(x; \varepsilon) = \frac{1}{e^{\lambda_+} - e^{\lambda_-}} \left[ e^{\lambda_+ x} - \left(1 + e^{\lambda_-}\right)e^{\lambda_+ x} - \left(1 + e^{\lambda_+}\right)e^{\lambda_- x} \right]. \]  

(2.2)

We remark for later use that, Taylor-expanding \( \lambda_\pm, e^{\lambda_\pm}, \text{ and } e^{\lambda_\pm x} \) and retaining all terms up to and including \( \mathcal{O}(\varepsilon), \) we find

\[ y(x; \varepsilon) = \frac{1}{e - e^{-1}} \left\{ 2 \left[ \sinh(x) + \sinh(x - 1) \right] + \varepsilon \left[ (x - 1) \sinh(x) + x \sinh(x - 1) \right] + \mathcal{O}(\varepsilon^2) \right\}. \]  

(2.3)

The same asymptotic result can be obtained by postulating the asymptotic expansion

\[ y(x) = \sum_{n \geq 0} \phi_n(\varepsilon) y_n(x) \]  

(2.4)

for the solution, substituting into the ODE (2.1), matching terms to derive ODEs for each one of the components \( y_0, y_1, \ldots, \) and solving these ODEs with the help of the boundary conditions to find these components.

- Substitution yields

\[ 0 = \sum_{n \geq 0} \phi_n y_n'' - \varepsilon \sum_{n \geq 0} \phi_n y_n' - \sum_{n \geq 0} \phi_n y_n = \left[ \phi_0 y_0'' + \phi_1 y_1'' + o(\phi_1) \right] - \varepsilon \left[ \phi_0 y_0' + \phi_1 y_1' + o(\phi_1) \right] - \left[ \phi_0 y_0 + \phi_1 y_1 + o(\phi_1) \right], \]

where we have assumed that the asymptotic expansion (2.4) can be differentiated term-by-term (recall that this is not possible, in general).

- The boundary conditions yield

\[ \sum_{n \geq 0} \phi_n(\varepsilon) y_n(0) = -1 \quad \text{and} \quad \sum_{n \geq 0} \phi_n(\varepsilon) y_n(1) = 1, \quad \text{whence} \quad \phi_0(\varepsilon) = 1 \text{ and} \left\{ y_0(0) = -1 \text{ and } y_0(1) = 1, \quad y_n(0) = y_n(1) = 0, \ n \geq 1. \right\} \]

- The only leading order terms that can be matched in the ODE above are those in the first and third terms, since \( \varepsilon \phi_0(\varepsilon) = o(\phi_0(\varepsilon)). \) Hence,

\[ y_0'' - y_0 = 0, \quad \text{whence} \quad y_0(x) = c_{+0} e^{x} + c_{-0} e^{-x}. \]  

(2.5)

- Employing the boundary conditions \( y_0(0) = -1 \) and \( y_0(1) = 1, \) we find

\[ y_0(x) = \frac{2}{e - e^{-1}} \left[ \sinh(x) + \sinh(x - 1) \right]. \]  

(2.6)

This concludes the determination of \( \phi_0 \) and \( y_0, \) and hence also of the leading behavior of \( y. \)

\[ ^1 \text{Note that, strictly speaking, the boundary conditions only yield that } \lim_{\varepsilon \to 0} \phi_0(\varepsilon) = 1 \text{—equivalently, that } \phi_0 = \mathcal{O}(1) \text{—and not that } \phi_0(\varepsilon) = 1. \text{ Since each member of an asymptotic sequence is characterized solely by its limit behavior for } \varepsilon \downarrow 0, \text{ we may choose } \phi_0(\varepsilon) = 1 \text{ without loss of generality. Note that this choice simplifies considerably the boundary conditions for the elements of } \{y_n\}_{n \geq 0}. \text{ [Work out the boundary conditions for } y_0 \text{ and } y_1 \text{ for the choices } \phi_0(\varepsilon) = 1/(1 + \varepsilon) \text{ and } \phi_1(\varepsilon) = \varepsilon/(1 + \varepsilon).] \]
• We now match the next-order terms. If $\phi_1(\varepsilon) \gg \varepsilon$, then the only possible matching is between $\phi_1 y_1''$ and $\phi_1 y_1$. The outcome is an ODE for $y_1$ identical to that for $y_0$ (see (2.5)) and equipped with homogeneous boundary conditions; it follows that $y_1 \equiv 0$, and hence the term $\phi_1 y_1$ is, in fact, absent from the asymptotic expansion. Therefore, the next-order correction must be at most $O(\varepsilon)$. If $\phi(\varepsilon) \ll \varepsilon$, now, then the next-order correction is $-\varepsilon \phi_0(\varepsilon)y_0' = -\varepsilon y_0'$. This cannot be matched by any other term, as these are asymptotically smaller, and thus it must be identically zero. We thus reach a contradiction, as $y_0' \neq 0$ by (2.6), and hence $\phi(\varepsilon) \not\ll \varepsilon$ either. We thus confidently choose $\phi(\varepsilon) = O(\varepsilon)$ (note that this scaling satisfies the aforementioned principle of least degeneracy, as all three terms—$\phi_1 y_1''$, $\varepsilon \phi_0 y_0'$, and $\phi_1 y_1$—now play a role). For simplicity, we let $\phi(\varepsilon) = \varepsilon$, whence

$$y_1'' - y_1 = y_0', \quad \text{with} \quad y_1(0) = y_1(1) = 0.$$  

(2.7)

This problem is explicitly solvable using variation of parameters,

$$y_1(x) = c_{+,1} e^x + c_{-,1} e^{-x} - e^x \int_0^x \frac{y_0'(s) e^{-s}}{W(e^s, e^{-s})} ds + e^{-x} \int_0^x \frac{y_0'(s) e^s}{W(e^s, e^{-s})} ds$$

$$= \left[ c_{+,1} + \frac{1 + e^{-1} x}{e - e^{-1}} \frac{1}{2} + \frac{1}{4} \right] e^x + \left[ c_{-,1} - \frac{1 + e}{e - e^{-1}} \frac{x}{2} - \frac{1}{4} \right] e^{-x},$$

where

$$W(e^s, e^{-s}) = \begin{vmatrix} e^s & e^{-s} \\ (e^s)' & (e^{-s})' \end{vmatrix} = -2$$

is the Wronskian of the fundamental solutions $e^x$ and $e^{-x}$.

• Employing the homogeneous boundary conditions, we obtain

$$y_1(x) = \frac{1}{e - e^{-1}} \left[ (x - 1) \sinh(x) + x \sinh(x - 1) \right].$$  

(2.8)

This process can be repeated as many times as needed to obtain higher order terms in the asymptotic expansion of the solution $y(\cdot; \varepsilon)$.

**Discussion.** Note that (2.6) and (2.8) match the two-term Taylor expansion (2.3): in fact, it can be shown that the asymptotic expansion for $y$ matches the Taylor expansion of the exact solution (2.2) to all orders in $\varepsilon$. A corollary of this observation is that (2.4) can, indeed, be differentiated term-by-term, since it is convergent for $t \in \mathbb{R}$, and $\varepsilon$ in a neighborhood of zero. Additionally, the $O_\varepsilon(\varepsilon)$ component of our two-term expansion (2.3) remains $O_\varepsilon(\varepsilon)$ uniformly over any bounded subset of $\mathbb{R}$—a property which was advertised in Sect. 1.5 as characteristic of regularly perturbed problems.

A plot of the solution together with the associated phase plane analysis is given in Fig. 4. The effect of the perturbative term $\varepsilon y$ on any bounded subset of the phase plane—such as the one depicted in that figure—is limited to a (uniformly) $O_\varepsilon(\varepsilon)$—perturbation of the orbits and of their parameterizations by time.

2.2 A singularly perturbed ODE

[See [Kevorkian & Cole] Sect. 2.1 for another discussion of this example.] Consider, now, the damped linear oscillator with small mass,

$$m \frac{d^2Y}{dT^2} + b \frac{dY}{dT} + kY = 0, \quad \text{with} \quad Y(0) = 0 \quad \text{and} \quad \frac{dY(0)}{dT} = v_0.$$  

(2.9)

Here, $Y$ is the displacement of the oscillator from the equilibrium state, $m$ its (“small”) mass, $v_0$ its initial velocity, $b > 0$ the damping coefficient, and $k > 0$ the spring constant, see also Fig. 5.
Figure 4: Left panel: the exact, perturbed ($\varepsilon > 0$) solution to (2.1) as given in (2.2) (solid red) and plotted in the $(y, \dot{y})$–plane (phase plane) together with its extension outside the unit time interval (dashed red). Also shown: the exact, unperturbed ($\varepsilon = 0$) solution (solid black) together with its extension (dashed black); the perturbed (solid blue) and unperturbed (solid black) eigenspaces; several other orbits corresponding to the same ODE (dashed blue) but failing to satisfy the boundary conditions. The solid blue part of those orbits residing in the top quarter of the phase plane corresponds to the unit time interval: note that these orbits fail either because they are too close to the saddle point and thus too slow or because they are too far removed from that point and thus too fast. Right panel: the perturbed—$y(\cdot; \varepsilon)$, in red—and unperturbed—$y(\cdot; 0)$, in black—solutions plotted as functions of time.

Figure 5: Schematic depiction of the mass–damper–spring system described by (2.9).

2.2.1 Scaling the model

Our first task is to non-dimensionalize (scale) the independent ($T$) and dependent ($Y$) variables. To quote Segel and Slemrod [Segel & Slemrod], some of the considerations entering scaling are the following: the scale of a dependent variable should provide an estimate of the order of magnitude of that variable; the scale of an independent variable should provide an estimate of the range of that variable over which the dependent variables change significantly. A crucial third consideration is that ‘[...] scaling requires prior knowledge of the solution. This can be provided by experiments, “physical intuition,” and/or numerical analyses of special cases.’

\[\text{We leave out, for the time being, an equally crucial fourth consideration—namely, ‘It may be necessary to choose different scales in different domains of the independent variable. If so, the appropriate approximation methods are of singular perturbation type.’}\]— It will be inadvertently brought to the spotlight in one of the later sections.
We begin to glean information on the solution by energy considerations. Initially, the mass is at equilibrium ($Y(0) = 0$) and thus the system only has kinetic energy, $E(0) = mv_0^2/2$. At the point of maximum displacement, $Y(T_s) = Y_M$, velocity—and thus also kinetic energy—is zero and the spring has stored potential energy equal to $E(T_s) = kY_M^2/2$. By conservation of energy,

$$\frac{1}{2}mv_0^2 = \frac{1}{2}kY_M^2 + W_d, \quad \text{where } W_d = \int_0^{Y_M} b \frac{dY(T)}{dT} dY(T) = \int_0^{T_s} b \left( \frac{dY(T)}{dT} \right)^2 dT > 0 \quad (2.10)$$

is the work done by the damping force (i.e., lost to friction). Hence,

$$\frac{1}{2}kY_M^2 < \frac{1}{2}mv_0^2, \quad \text{or equivalently, } Y_M < \sqrt{\frac{m}{k}}v_0.$$

If $v_0$ remains bounded as $m \downarrow 0$, then $Y_M \downarrow 0$ by the above inequality. In this example, we choose to examine motion the maximum displacement $Y_M$ of which remains bounded and nonzero as $m \downarrow 0$; hence, $v_0$ must scale at least as $m^{-1/2}$ for $Y_M$ to be nonzero; consequently, $v_0$ must be unbounded as $m \downarrow 0$.

A first consequence of this result is that the damping force $b dY/dT$ initially dominates over the restoring force $kY$: indeed, $b dY/dT \ll m^{-1/2}$ while $kY \gg 1$, since $Y$ ranges in the bounded interval $[0, Y_M]$. Hence, one expects that the last term in the left member of (2.9) may be omitted “to leading order” to obtain the approximate ODE

$$m \frac{d^2Y'}{dT^2} + b \frac{dY'}{dT} = 0, \quad \text{with } Y(0) = 0 \text{ and } \frac{dY(0)}{dT} = v_0. \quad (2.11)$$

The solution of this approximate problem is

$$Y(T) = \frac{mv_0}{b} \left( 1 - e^{-bT/m} \right), \quad (2.12)$$

a fact which suggests the scalings $\psi = Y/|Y|$ and $\tau = T/|T|$, with $|Y| = mv_0/b$ and $|T| = m/b$.

Note that, under this scaling, $v_0$ must scale like $m^{-1} \gg m^{-1/2}$ in order to obtain an $O_s(1)$ maximum displacement $Y_M = mv_0/b$! This apparent discrepancy is due to the fact that, for $v_0 \gg 1$, essentially all kinetic energy is lost to friction and only an $o(E(0))$ amount is left to be stored in the spring. Indeed, the amount of energy lost to friction can be determined by (2.10), if $T_s$ is known. Note that (2.12) predicts an exponential approach to $Y = Y_M$, i.e., that the mass will never reach its maximum displacement but, rather, tend to it asymptotically. The reason for this discrepancy between our intuition and our analytic formula is our omission in (2.11) of the restoring force $kY$, which becomes commensurate to damping as the velocity decreases and the displacement approaches its maximum. Hence, we will estimate $T_s$ by demanding that the damping force becomes $O_s(1)$, as the restoring force is near $Y_M$. Using (2.12), we calculate $dY(T)/dT = v_0 e^{-bT/m}$, and hence the velocity becomes $O_s(1)$ for $e^{-bT_s/m} = O_s(v_0^{-1}) = O_s(m) = o(1)$. (We remark, for later use, that this entails the asymptotic relation $T_s = O_s(m \ln m)$, and hence $T_s \downarrow 0$ as $m \downarrow 0$.) Using this estimate and (2.10), we calculate

$$W_d = \int_0^{Y_M} b \left( \frac{dY(T)}{dT} \right)^2 dT = \int_0^{T_s} b v_0^2 e^{-2bT_s/m} dT = E(0) \left( 1 - e^{-2bT_s/m} \right),$$

with $E(0) = mv_0^2/2$ as before. The amount of energy left to be stored in the spring equals $E(0) e^{-2bT_s/m}$ which is, indeed, asymptotically smaller than $E(0)$ (by our estimate $e^{-2bT_s/m} = O(1)$ above).

Using the scalings $\psi = bY/(mv_0)$ and $\tau = bT/m$ introduced above, we rewrite (2.9) in the form

$$\psi''(\tau; \varepsilon) + \psi'(\tau; \varepsilon) + \varepsilon \psi(\tau; \varepsilon) = 0, \quad \text{with } \psi(0; \varepsilon) = 0 \text{ and } \psi'(0; \varepsilon) = 1, \quad \text{and where } \varepsilon = \frac{mk}{b^2} > 0 \quad (2.13)$$

is a parameter taking values in a neighborhood of zero, on account of $m$ being allowed to become arbitrarily small. Additionally, $\psi' = d \cdot d\psi$. Note, also, that our scaling has brought down the number of parameters present in the model from four ($m$, $b$, $k$, and $v_0$) to one ($\varepsilon$); this feat alone is justification enough for the effort we put into scaling the model, as it reveals that systems satisfying (2.9) but equipped with different sets of parameters may exhibit similar behavior. In particular, the four-dimensional parameter space is foliated: each foil is defined by the relation $\varepsilon = mk/b^2 = const.$, and parameter values belonging to the same foil yield systems with identical behavior up to rescaling time and/or space.
2.2.2 The initial transient

The problem presented in (2.13) is regularly perturbed, as setting \( \varepsilon \) to zero does not cause a reduction in the order of the ODE. Following our work in Sect. 2.1, we postulate the asymptotic expansion

\[
\psi(\tau; \varepsilon) = \sum_{n \geq 0} \phi_n(\varepsilon)\psi_n(\tau),
\]

and substitute into (2.13) to find

\[
\sum_{n \geq 0} \phi_n\psi_n'' + \sum_{n \geq 0} \phi_n\psi_n' + \varepsilon \sum_{n \geq 0} \phi_n\psi_n = 0, \quad \text{with} \quad \sum_{n \geq 0} \phi_n(\varepsilon)\psi_n(0) = 0 \quad \text{and} \quad \sum_{n \geq 0} \phi_n(\varepsilon)\psi_n'(0) = 1,
\]

and group terms of equal asymptotic order to find

\[
\psi_0'' + \psi_0' = 0, \quad \text{with general solution} \quad \psi_0(\tau) = C_{+0} + C_{-0} e^{-\tau}.
\]  

(2.15)

Now, the initial condition pertaining to the velocity yields \( \phi_0(\varepsilon) = 1 \)—in fact, we could have guessed this readily from our scaling analysis above, as the approximate solution (2.12) and our scaling for \( Y \) suggest that \( \psi \) lies in the unit interval, in the timescale under examination. Hence, we may select \( \phi_0 \equiv 1 \) and thus recast the boundary conditions as

\[
\forall n \geq 0 [\psi_n(0) = 0], \quad \psi_0'(0) = 1, \quad \text{and} \quad \forall n \geq 1 [\psi_n'(0) = 0].
\]

Combining the boundary condition for \( \psi_0 \) with (2.15), we then find

\[
\psi_0(\tau) = 1 - e^{-\tau}.
\]  

(2.16)

Higher order terms may be obtained in a manner similar to that we employed in Sect. 2.1. For example, employing the principle of least degeneracy to set \( \phi_1(\varepsilon) = \varepsilon \) (the reader can exclude the cases \( \phi_1 \ll \varepsilon \) and \( \phi_1 \gg \varepsilon \) as in the aforementioned section), we find the linear, inhomogeneous problem

\[
\psi_1'' + \psi_1' = -\psi_0, \quad \text{with} \quad \psi_1(0) = 0 \quad \text{and} \quad \psi_1'(0) = 0.
\]

The solution to this problem is \( \psi_1(\tau) = 2\tau + (2 + \tau)e^{-\tau} \), so that the two-term expansion for \( \psi \) is

\[
\psi(\tau; \varepsilon) = 1 - e^{-\tau} + \varepsilon \left[ 2\tau - (2 + \tau)e^{-\tau} \right] + \mathcal{O}_3(\varepsilon^2).
\]  

(2.17)

As we discussed earlier, the leading order term in the truncated asymptotic expansion (2.17) predicts exponential decay towards \( \psi = 1 \). This picture is significantly altered by the next-order term in that expansion, which grows unboundedly with \( \tau \) due to the linear term in it. The expansion only remains well-ordered for \( \tau = o(1/\varepsilon) \); for larger values of \( \tau \), the second term in the asymptotic expansion becomes commensurate with the first; in fact, this is true of all subsequent terms, as (it is easy to show that) the fastest-growing term in \( \psi_n(\tau) \) grows as \( \tau^n \).

Recalling the definitions of \( \varepsilon \) and \( \tau \), we find that the relation \( \tau = o(1/\varepsilon) \) becomes \( kT/b = o(1) \) \((m \downarrow 0)\)—that is, one can expect that the asymptotic expansion only provides a legitimate solution to the problem over a time interval \([0, T]\), with \( T \downarrow 0 \) as \( m \downarrow 0 \) (The meaning of the non-dimensional time \( kT/b \) appearing above will be clarified in the next section.) Past that short time interval, our assumption that the restoring force is higher-order becomes invalid and we have to use a different scaling. This fact lends this initial behavior of the system its name—transient—as it soon offers its place to phenomena of an entirely different qualitative nature.

2.2.3 Behavior past the initial transient

To describe the behavior of the system past the initial transient analyzed above, we zoom out and consider a longer timescale. Since the asymptotic expansion derived for (2.13) breaks down for \( \tau = o(1/\varepsilon) \), we will focus precisely on \( \mathcal{O}_3(1/\varepsilon) \) —values of \( \tau \). To that effect, we rescale \( \tau \) and define the new time variable \( t = \varepsilon \tau = kT/b \), writing also \( \psi(\tau; \varepsilon) = \psi(t/\varepsilon; \varepsilon) = y(t; \varepsilon) \) for clarity. Under this rescaling, (2.13) becomes

\[
\varepsilon \ddot{y}(t; \varepsilon) + \dot{y}(t; \varepsilon) + y(t; \varepsilon) = 0, \quad \text{with} \quad y(0; \varepsilon) = 0 \quad \text{and} \quad \dot{y}(0; \varepsilon) = \frac{1}{\varepsilon},
\]  

(2.18)

\footnote{The attentive reader will realize that this is merely the rescaled form of (2.12), just as the problem (2.15) it satisfies is merely the rescaled version of (2.11).}
and where we have written \( \dot{\psi} = d \cdot /dt \). (Note that (2.18) is of singular perturbation type, since setting \( \varepsilon = 0 \) in it reduces the order of the ODE.) Here also, we postulate an asymptotic expansion
\[
y(t; \varepsilon) = \sum_{n \geq 0} f_n(\varepsilon)y_n(t),
\]
substitute into (2.13) to find
\[
\varepsilon \sum_{n \geq 0} f_n \ddot{y}_n + \sum_{n \geq 0} f_n \dot{y}_n + \sum_{n \geq 0} f_n y_n = 0,
\]
and group terms of like asymptotic order to find
\[
\dot{y}_0 + y_0 = 0, \quad \text{with general solution } y_0(t) = C_0 e^{-\varepsilon t}.
\]
Here, \( C_0 \) is an arbitrary constant. Higher order terms may be obtained in a similar manner. For example, at next order, we set \( f_1(\varepsilon) = \varepsilon f_0(\varepsilon) \) to find the linear, inhomogeneous problem
\[
\dot{y}_1 + y_1 = -\dot{y}_0, \quad \text{with general solution } y_1(t) = (C_1 - C_0 t) e^{-\varepsilon t}.
\]
Here, \( C_1 \) is another arbitrary constant. Hence, the two-term expansion for \( y \) is
\[
y(t; \varepsilon) = C_0 e^{-\varepsilon t} + \varepsilon (C_1 - C_0 t) e^{-\varepsilon t} + O_\varepsilon(\varepsilon^2),
\]
with the constants \( C_0 \) and \( C_1 \) remaining undetermined.

At this point, it is not clear how these constants must be selected, as only one initial condition can be accommodated. The initial condition \( y(0; \varepsilon) = 0 \) yields \( y_0(t) \equiv 0 \), which implies that the displacement \( y \) in this timescale is \( o(1) \)—this is in disagreement with our intuition on the behavior of the system. The initial condition \( \dot{y}(0; \varepsilon) = 1/\varepsilon \), on the other hand, yields \( f_0(\varepsilon)y_0(t) = -\varepsilon^{-1} e^{-\varepsilon t} \); this is equally unrealistic, as it implies that \( y \gg 1 \) and, also, that \( y \) is negative (!) although \( y(0; \varepsilon) = 0 \) and \( \dot{y}(0; \varepsilon) > 0 \). In truth, we should not use any of the prescribed initial conditions, as we do not expect (2.19) to describe the behavior of the system as \( t \downarrow 0 \). To see this, recall that the behavior of the system close to zero is described, to leading order, by \( \psi(\tau; \varepsilon) \sim -1 - e^{-\tau} \). Passing to \( t = \tau/\varepsilon \), we rewrite this as \( \dot{y}(t; \varepsilon) \sim 1 - e^{-t/\varepsilon} \); note that the exponential term in this relation is exponentially small for each \( t > 0 \) but not uniformly over any time interval containing zero. In fact, this relation suggests that
\[
\dot{y}(t; \varepsilon) \sim -\varepsilon^{-1} e^{-t/\varepsilon} \quad \text{and} \quad \ddot{y}(t; \varepsilon) \sim -\varepsilon^{-2} e^{-t/\varepsilon},
\]
and hence \( \varepsilon \ddot{y} \) in (2.18) is not higher-order but, instead, of the same order as \( \dot{y} \). (This agrees with our analysis in the last section, naturally.) An asymptotic expansion such as the one in (2.19), on the other hand, could never capture this behavior, as the second-order term in (2.18) ends up being considered higher-order. We will see how to determine the arbitrary constants in (2.21) through matching in the next section.

### 2.2.4 Matching

The idea behind matching is that the solutions to (2.13) and (2.18) represent the same, unique solution in different timescales. Since \( \psi \) is valid in a neighborhood of zero which vanishes as \( \varepsilon \downarrow 0 \) and \( y \) is valid everywhere except for a neighborhood of zero, it seems natural to try to identify a region where both are equally valid. Should we manage to locate such a region, the solutions should match (i.e., be identical) in it. In this case, we will be able to use our explicit knowledge of (2.14) to determine the unknown constants in (2.19).

First, we deal with the leading order results (2.16) and (2.20) for the inner solution \( \psi \) and the outer solution \( y \), respectively. (The terms ‘inner’ and ‘outer’ reflect the presence of a boundary layer for our problem: the ‘inner solution’ is the solution inside that boundary layer, while the ‘outer solution’ is the solution outside it.) As discussed earlier, \( \lim_{\tau \to \infty} \psi_0(\tau) = 1 \); additionally, \( \lim_{t \downarrow 0} y_0(t) = C_0 \). Continuity of the solution implies, then, that \( C_0 = 1 \). In fact, we can add more content to this fast calculation by showing that there is an entire range of intermediate timescales—all of them slower than the fast one and faster than the slow one—in which the two solutions \( \psi_0 \) and \( y_0 \) match exactly, for this specific choice of \( C_0 \). To locate them, let \( \delta : I \to \mathbb{R} \) be a function such that \( \varepsilon \ll \delta \ll 1 \), and introduce the variable \( s = \delta \tau = \delta t/\varepsilon \). By virtue of \( \varepsilon \ll \delta \ll 1 \), \( O_\varepsilon(1) \)—values
of $s$ correspond to asymptotically large values of $\tau$ and asymptotically small values of $t$. Recasting (2.16) and (2.20) in terms of this new variable, we find, for $s$ in any fixed interval $[\underline{s}, \bar{s}]$ (with $\underline{s} > 0$),

$$\psi_0(\tau(s)) = 1 - e^{-s/\delta} = 1 + O\left(e^{-s/\delta}\right) \quad \text{and} \quad y_0(t(s)) = C_0 e^{-\varepsilon s/\delta} = C_0 + O\left(\varepsilon/\delta\right).$$

Indeed, then, the two formulas match each other exactly, as long as $C_0 = 1$. This match is extended to all intermediate timescales, i.e., for all $\varepsilon \ll \delta \ll 1$.

This result can be extended to higher orders. Indeed, recasting the two-term expansion (2.17) for $\psi$ in terms of $s$, we obtain

$$1 - e^{-s/\delta} + \varepsilon \left[2 - \frac{s}{\delta} - \left(2 + \frac{s}{\delta}\right) e^{-s/\delta}\right] = 1 + \varepsilon \left[2 - \frac{s}{\delta}\right] + O_s(\varepsilon^2/\delta^2),$$

since the exponential term is asymptotically smaller than all algebraic terms. Note that the $O_s(\varepsilon^2/\delta^2)$—remainder derives from $\varepsilon^2 \psi_2$—the $O_s(\varepsilon^2)$—term in the asymptotic expansion of $\psi$. (Here, we have used our earlier remark on the fastest-growing term in $\psi_n$.) The two-term expansion (2.21) for $y$, in turn, becomes

$$C_0 e^{-\varepsilon s/\delta} + \varepsilon \left[C_1 - C_0 \frac{\varepsilon s}{\delta}\right] e^{-\varepsilon s/\delta} = C_0 + \varepsilon \left[C_1 - C_0 \frac{s}{\delta}\right] + O_s(\varepsilon^2/\delta^2),$$

where we have Taylor-expanded the exponential terms, as their arguments are asymptotically small. The two formulas match as long as $C_1 = 2$, which fixes the value for the second arbitrary constant, too.

We finally note that, following matching, the two (truncated) asymptotic expansions above can be combined to derive a uniformly valid (truncated) asymptotic expansion. Indeed, adding them up and subtracting their common part at each order, as this was identified above, we find

$$e^{-t} + \varepsilon(2 - t)e^{-t} - e^{-t/\varepsilon} + (2\varepsilon + t)e^{-t/\varepsilon} + O_s(\varepsilon^2).$$

(2.22)

### 2.2.5 Discussion

Our results in Sect. 2.2.2 and 2.2.3 exemplify Segel and Slemrod’s admonition on the necessity of choosing different timescales to describe different phases of the system evolution. Indeed, to elucidate the behavior of the system during the short initial transient, we have worked with the fast time $\tau = bT/m$. In that phase, friction dominates and causes a radical decrease in momentum; hence, the behavior of the system is described, to leading order, by (2.11)—or, in its rescaled form, (2.15). To analyze the behavior in the much slower timescale, we have recast the system in the slow time $t = \bar{T}/m$. In this slow phase, friction and the restoring force are both moderate and approximately counteract each other, so that the momentum barely changes. To leading order, then, the system evolution is described by

$$b \frac{dY}{dT} + kY = 0,$$

which is the dimensional form of (2.20). Note that this ODE is simply obtained by omitting the inertial term $mY''$ in (2.9)—a “natural” choice, as $m$ is assumed “small”, albeit one that only becomes effective once the fast initial transient plays itself out.

Another focal point of our work in this section was non-dimensionalizing the original system (2.9). It is worth remarking here that, since the ODE satisfied by $Y$ is linear, it remains invariant under all scalings $y = Y/|Y|$ (the prefactors of $|Y|$ ‘cancel out’). This is hardly surprising, of course, as the solutions to linear equations form a vector space and thus there can be no “natural” scaling associated with equations of that type. It is the initial or boundary conditions that select a unique solution, and hence it is these conditions that determine a natural scaling. In our work here, the scaling $|Y|$ was suggested by the condition we imposed that the amplitude of the motion should remain bounded—both above and below—as the mass $m$ of the body limits down to zero; the actual scaling was then derived using the initial conditions and physical intuition on the relative magnitudes of various terms in the ODE.

Regarding matching, we note that the two-term uniform asymptotic expansion in (2.22) is, in this particular example, the (multiscale) Taylor expansion of the exact solution,

$$y(t; \varepsilon) = \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\varepsilon (\lambda_+ - \lambda_-)}, \quad \text{where} \quad \lambda_\pm = \frac{-1 \pm \sqrt{1 - 4\varepsilon}}{2\varepsilon}.$$  

(2.23)
Figure 6: Left panel: the exact, perturbed ($\varepsilon > 0$) solution to (2.9) as given in (2.23) (solid red) and plotted in the phase plane. Note the points corresponding to the time instants $t_*$ and $t_{**}$, at which $y(t_*) = 0$ (vertical vector field) and $\dot{y}(t_{**}) = 0$ (horizontal vector field)—cf. problem 2(b–c) in Homework #02. Also shown: several other orbits corresponding to the same ODE (dashed blue) but failing to satisfy the initial conditions; and the perturbed (solid blue) and unperturbed (solid black) eigenspaces. Right panel: the perturbed solution $y(\cdot; \varepsilon)$ plotted as functions of time, together with the points corresponding to the time instants $t_*$ (the tangent is horizontal) and $t_{**}$ (the tangent osculates the graph of the solution) at the edge of the boundary layer.

are the corresponding eigenvalues. (See also Fig. 6, where the solution is plotted both in the corresponding phase space and as a time trajectory.) To see this, note that

$$\lambda_+ = -1 - \varepsilon + \mathcal{O}(\varepsilon^2) \quad \text{and} \quad \lambda_- = -\frac{1}{\varepsilon} + 1 + \varepsilon + \mathcal{O}(\varepsilon^2).$$

Substituting, now, into (2.23), Taylor-expanding the first exponential, and omitting the second exponential, we arrive at (2.21) with $C_0 = 1$ and $C_1 = 2$—recall that matching led to the same values for these constants. We remark that the omission of $e^{\lambda_+ t}$ is valid as long as $t \gg \varepsilon$, as this term is, then, asymptotically smaller than all algebraic terms—recall that this condition was also employed in matching, while discussing the properties of the intermediate timescale. Switching, on the other hand, to the fast variable $\tau = t/\varepsilon$, we obtain the solution $\psi(\tau; \varepsilon) = y(\tau; \varepsilon)$. In this case, the first exponential can be Taylor-expanded around zero as long as $\tau \ll 1/\varepsilon$ (equivalently, $t \ll 1$), since then $\lambda_+ \tau \ll 1$. The exponent of the second exponential, on the other hand, is $\mathcal{O}(1)$ for values of $\tau$ covering an $\mathcal{O}(1)$ range, and thus must be retained as is. The result is, here again, (2.17).
01. Consider the regularly perturbed problem

\[ \ddot{y}(t) + (1 + \varepsilon f(t)) y(t) = 0, \]  
with \( y(0) = 0 \) and \( \dot{y}(0) = 1 \).

Here, \( 0 < \varepsilon \ll 1 \) as usual.

(a) Derive a two-term asymptotic expansion \( \phi_0(\varepsilon)y_0(t) + \phi_1(\varepsilon)y_1(t) \) for the solution \( y(t; \varepsilon) \) to this problem in the case \( f(t) = e^{-t} \). Is this expansion uniformly valid in \( \mathbb{R}^+ \)? Could you have foreseen this from the nature of the perturbation?

(b) Consider, now, the same problem with \( f(t) = 1 \). Derive, here also, a two-term asymptotic expansion for the solution to this problem, and show that \( \phi_1y_1 \) is not uniformly asymptotically smaller than the leading order term \( \phi_0y_0 \) over \( \mathbb{R}^+ \). Explain where this non-uniformity comes from.

(c*) Formulate a condition for \( f \) which is sufficient for \( \phi_1y_1 \ll \phi_0y_0 \) to hold uniformly over \( \mathbb{R}^+ \). Explain the meaning of that condition in physics terms.

02. Consider the rescaled problem (2.13)

\[ \psi''(\tau) + \psi'(\tau) + \varepsilon \psi(\tau) = 0, \]  
with \( \psi(0) = 0 \) and \( \psi'(0) = 1 \),

in which mass, damping coefficient, and initial velocity have been rescaled to one and the spring constant has been rescaled to \( \varepsilon \ll 1 \).

(a) Use the asymptotic expansion (2.14), with as many terms as needed, to derive a leading order formula for the time instant \( \tau_* \) at which \( \psi'(\tau_*) = 0 \).

(b) Using this leading order formula for \( \tau_* \), calculate the work done by friction over the time interval \([0, \tau_*] \) and up to and including terms of \( O_s(\varepsilon) \). Additionally, estimate the energy stored in the spring to leading order. Do the two results imply conservation of energy up to and including terms of \( O_s(\varepsilon) \)? Explain.

(c*) Repeat the same procedure as in (a) to derive a leading order formula for the time instant \( \tau_{**} \) at which \( \psi''(\tau_{**}) = 0 \). Show that, while the point \((\psi(\tau_*), \psi'(\tau_*))\) in the phaseplane is an \( O_s(1) \) distance from the eigenspace corresponding to the \( O_s(\varepsilon) \) eigenvalue (slow eigenspace), the point \((\psi(\tau_{**}), \psi'(\tau_{**}))\) is only \( O_s(\varepsilon) \) away from it. [Note that this generalizes to \( n \)th order derivatives, thus yielding a method to approximate points on that slow eigenspace.]

03. [Kevorkian & Cole, Sect. 2.2, Ex. 2] Consider the boundary-value problem

\[ \varepsilon y'' + \frac{1}{\sqrt{x}} y' - y = 0, \]  
with \( y(0) = 0 \) and \( y(1) = e^{2/3} \), and where \( (\cdot)' = \frac{d}{dx} \).

(a) Calculate a leading order (as \( \varepsilon \downarrow 0 \)) approximation to the solution of this problem which is uniformly valid over the interval \([0, 1] \).

(b*) Can you interpret your results geometrically using a plot of the dynamics in the \((x, y, y')\) space (extended phase space)?
3.1 The linear problem

Seeing as our work in the preceding two lectures has been confined to linear models, we briefly present an overview of singularly perturbed such models to introduce notation and concepts. Let us consider, then, the linear, constant-coefficient problem in $\mathbb{R}^N$

$$w' = A^\varepsilon w, \quad \text{where } A^\varepsilon \text{ is an } \varepsilon-\text{dependent, constant } N \times N \text{ matrix and } w = (w_1, \ldots, w_N)^T. \quad (3.1)$$

Here, $(\cdot)' = d/\,d\tau$ denotes differentiation with respect to time, measured by the independent variable $\tau$. The superscript $\varepsilon$ in $A^\varepsilon$ denotes the dependence of that matrix on the small parameter; it is not a power. The same is true of all superscripts below, with the exception of $(\cdot)^{-1}$ which denotes the inverse of the parenthesized matrix.

3.1.1 Basic assumptions and definitions

Assume that we know the following for $N \times N$ matrix $A^0 = A^\varepsilon = 0$:

- It possesses a zero eigenvalue of algebraic and geometric multiplicity equal to $n < N$.
- All other eigenvalues of $A^0$ may be complex but are bounded away from the imaginary axis $i\mathbb{R}$; that is, they all have nonzero real parts.

Alongside these assumptions, let us define certain quantities of interest:

- We denote by $U^0$ the eigenspace of $A^0$ associated with the zero eigenvalue—i.e., $U^0 = \text{Ker}(A^0)$—and let $\{u_1^0, \ldots, u_n^0\}$ be any basis of vectors in $\mathbb{R}^N$ for that space. That is, $U^0 = \text{span}\{u_1^0, \ldots, u_n^0\}$ and each basis vector is mapped to zero by the matrix, $A^0 u_1^0 = \ldots = A^0 u_n^0 = 0$.
- We write $\lambda_1^0, \ldots, \lambda_m^0$ (with $m \leq \ell := N - n$) for the remaining eigenvalues. Without loss of generality, we arrange these in non-descending order of their real parts:

$$\text{Re}(\lambda_1^0) \leq \ldots \leq \text{Re}(\lambda_m^0) < 0 < \text{Re}(\lambda_{m+1}^0) \leq \ldots \leq \text{Re}(\lambda_m^0), \quad \text{for some } 0 \leq m_- \leq m.$$

The case $m_- = 0$ is taken to mean that all eigenvalues have positive real parts (are unstable); the case $m_- = m$ corresponds to all eigenvalues having negative real parts (being stable); and any two eigenvalues are allowed to have the same real part only if they are complex conjugates (that is, we do not list eigenvalues according to their multiplicities). We also introduce the generalized eigenvectors $v_0^1, \ldots, v_{n_1}^1$ for $\lambda_1^0$, $v_{n_1+1}^0, \ldots, v_{n_2}^0$ for $\lambda_2^0$, and so on all the way through $v_{n_{m-1}+1}^0, \ldots, v_{n_m}^0$ for $\lambda_m^0$; here, $n_1 < \ldots < n_m = \ell$.

- We further denote by $V^0 = \text{span}\{v_0^1, \ldots, v_{n_{m_-}}^1\}$ the $\ell$-dimensional, invariant under the action of $A^0$ subspace of $\mathbb{R}^N$ associated with these eigenvalues. We similarly write $V^-_m$ for the invariant eigenspace associated with the stable eigenvalues, which is of (some) dimension $\ell_- = n_{m_-} \geq m_-$ and spanned by $\{v_{n_1}^0, \ldots, v_{n_{m_-}}^0\}$.

1Clearly, the numbers $n_1, n_2 - n_1, \ldots, n_m - n_{m_-}$ are the algebraic multiplicities of the eigenvalues.
Given these definitions, we can write $\mathbb{R}^N = U^0 \oplus V^0_\bot \oplus V^0_\bot$. That is, every vector $w \in \mathbb{R}^N$ has a unique decomposition of the form $w = u + v_- + v_+$, with $u \in U^0$, $v_- \in V^0_\bot$ and $v_+ \in V^0_\bot$. Using the bases for these three spaces, we can write

$$u = \sum_{1 \leq i \leq n} x_i u_i^0, \quad v_- = \sum_{1 \leq i \leq \ell_-} y_i v_i^0, \quad \text{and} \quad v_+ = \sum_{1 \leq i \leq \ell_+} y_i v_{i+\ell_-}^0.$$  

This decomposition amounts to a change of coordinates $w \mapsto z$, written more compactly in the form

$$w = P^0 z, \quad \text{where} \quad w = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}, \quad z = \begin{bmatrix} x \\ y_- \\ y_+ \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ y_1 \\ \vdots \\ y_{\ell_-} \\ y_{\ell_-+1} \\ \vdots \\ y_{\ell_+} \end{bmatrix} \quad \text{and} \quad P^0 = \begin{bmatrix} u_1, \ldots, u_n, v_1^0, \ldots, v_\ell^0 \end{bmatrix} = [U^0, V^0_-, V^0_+] .$$

(Note that $P^0$ here is an $N \times N$ matrix with its columns listed explicitly; the expressions for the $N \times n$, $N \times \ell_-$ and $N \times \ell_+$ matrices $U^0$, $V^0_-$ and $V^0_\ell$ are hopefully clear.) The inverse change of coordinates $z \mapsto w$ reads

$$z = (P^0)^{-1} w, \quad \text{where} \quad (P^0)^{-1} = \begin{bmatrix} U_+^{1,0} & V_+^{1,0} & V_+^{1,0} \\ U_{++}^{1,0} & V_{++}^{1,0} & V_{++}^{1,0} \\ U_{+-}^{1,0} & V_{+-}^{1,0} & V_{+-}^{1,0} \end{bmatrix} ; \quad \text{hence}, \quad z = \begin{bmatrix} x \\ y_- \\ y_+ \end{bmatrix} = \begin{bmatrix} V_+^{1,0} w \\ U_{++}^{1,0} w \\ U_{+-}^{1,0} w \end{bmatrix} .$$

The notation here is suggestive (if somewhat cluttered): the columns of the $N \times n$ block $U^0$ span the space $U^0$ and those of the $N \times \ell_-$ blocks $V^0_\bot$ span $V^0_\bot$. The rows, on the other hand, of the $n \times N$ block $V^{1,0} \bot$ span the orthogonal complement of $V^0_\bot$—written $(V^0_\bot)^{\perp}$—and those of the $\ell_+ \times N$ blocks $U^{1,0} \bot$ span the spaces $(U \oplus V^0_\bot)^{\perp}$. This becomes evident by writing out blockwise the identity $P^{-1} P = I$, \begin{equation}
\begin{bmatrix}
V^{1,0} U^0 & V^{1,0} V^0_- & V^{1,0} V^0_+

U^{1,0} U^0 & U^{1,0} V^0_- & U^{1,0} V^0_+

U^{1,0} U^0 & U^{1,0} V^0_- & U^{1,0} V^0_+
\end{bmatrix} = \begin{bmatrix}
I & 0 & 0
0 & I & 0
0 & 0 & I
\end{bmatrix},
\end{equation}

and interpreting the equations involving zero blocks as orthogonality relations.

### 3.1.2 The $\varepsilon = 0$ dynamics

Using this information, we can understand the dynamic behavior of our $\varepsilon = 0$ problem. A standard calculation yields, for $\varepsilon = 0$,

$$z' = (P^0)^{-1} A^0 P^0 z, \quad \text{where} \quad (P^0)^{-1} A^0 P^0 = \begin{bmatrix}
V^{1,0} A^0 U^0 & V^{1,0} A^0 V^0_- & V^{1,0} A^0 V^0_+

U^{1,0} A^0 U^0 & U^{1,0} A^0 V^0_- & U^{1,0} A^0 V^0_+

U^{1,0} A^0 U^0 & U^{1,0} A^0 V^0_- & U^{1,0} A^0 V^0_+
\end{bmatrix} .$$

Let us now derive concrete expressions for the blocks in this matrix.

- First, the leftmost (block-)column of the matrix is identically zero by virtue of the identity $A^0 U^0 = 0$. Indeed, recall that the columns of $U^0$ are associated with the zero eigenvalue.

- Second, note for immediate use below that $A^0 V^0_\pm = V^0_\pm \Lambda^0_\pm$, for some $\ell_+ \times \ell_\pm$ matrices $\Lambda^0_\pm$, because $V^0_\pm$ is invariant under the action of $A^0$ and associated with the unstable/stable eigenvalues. Since we selected the columns of $V^0_\pm$ to be generalized eigenvectors, $\Lambda^0_\pm$ will be in Jordan canonical form with the unstable/stable eigenvalues along its diagonal arranged according to their algebraic multiplicity.

\[\text{The reader should convince themselves that } A^0 V^0_\pm = V^0_\pm \Lambda^0_\pm \text{ is indeed an expression of invariance of the spaces spanned by the columns of } V^{0,0}_\pm.\]
both in forward and backward time, so that we can write $V_+^0 A^0 \Pi^0_+ = V_+^0 \Pi^0_+ A_+^0 = 0$, where the last step follows from (3.2).

By a similar token, the two remaining off-diagonal blocks are also zero, $U_\pm^0 A^0 \Pi^0_\pm = U_\pm^0 \Pi^0_\pm A^0_\pm = 0$.

Finally, the bottom diagonal blocks become $U_\pm^0 A^0 \Pi^0_\mp = U_\pm^0 \Pi^0_\mp A^0_\mp = \Lambda_\mp^0$.

It follows from the above that the $\varepsilon = 0$ ODE system for the $\ell$–components reads

$$\begin{align*}
x' &= 0, \\
y'_- &= \Lambda^-_0 y_-, \\
y'_+ &= \Lambda^+_0 y_+.
\end{align*}$$

(3.3)

The dynamic behavior of (3.3) is easy to describe in its entirety. First, the fixed points of this system are precisely the points on the $x$–coordinate space ($y = 0$), since the matrices $\Lambda^0 \pm$ are invertible. we will be writing $\mathcal{M}^0 = \{(x, 0) \mid x \in \mathbb{R}^\ell\}$ for this subspace of fixed points. Next, for every fixed $x \in \mathbb{R}^\ell$, the affine linear space $\mathcal{F}_0(x) = \{(x, y) \mid y \in \mathbb{R}^n\}$ is invariant under the flow generated by (3.3). This follows trivially by $x' = 0$, and the space is called an $\varepsilon = 0$ fast fiber. Finally, the subspaces $\mathcal{M}^0$ and $\mathcal{F}_0(x)$ intersect transversally (in fact, orthogonally) at the point $(x, 0)$—called the base point of that fiber—which is the unique fixed point on the invariant set $\mathcal{F}_0(x)$. In particular, the phase space $\mathbb{R}^{n+\ell}$ is foliated by these fast fibers,

$$\mathbb{R}^{n+\ell} = \bigcup_{(x, 0) \in \mathcal{M}^0} \mathcal{F}_0(x).$$

For each $x$, the dynamic behavior on $\mathcal{F}_0(x)$ is dictated by the ODE system $y'_\pm = \Lambda^0_\pm y_\pm$, which makes no reference to $x$. Hence, all fibers have identical dynamics. The evolution on a fiber $\mathcal{F}_0(x)$ is explicitly determined by integration of the ODEs for $y$. Writing $\phi^0_\ell(x, y)$ for the flow on that plane carrying the initial condition $(x, y)$ to the corresponding solution at time $\tau$, we find

$$\phi^0_\ell(x, y) = \begin{pmatrix} x \\ e^{\tau \Lambda^-_0} y_- \\ e^{\tau \Lambda^+_0} y_+ \end{pmatrix}.$$  

(3.4)

It follows that

$$\left\|e^{\tau \Lambda^-_0} y_-\right\| < C_- e^{-\tau \sigma_-} \left\|y_-\right\| \quad \text{(for } \tau > 0\text{)} \quad \text{and} \quad \left\|e^{\tau \Lambda^+_0} y_+\right\| < C_+ e^{\tau \sigma_+} \left\|y_+\right\| \quad \text{(for } \tau < 0\text{)},$$

for some constants $C_\pm \geq 1$, $0 < \sigma_- < \left|\text{Re} \Lambda^-_0\right|$ and $0 < \sigma_+ < \left|\text{Re} \Lambda^+_0\right|$. In other words, the $y_-$ and $y_+$–components of the solution contract exponentially in forward and backward time, respectively. In particular, initial conditions on an affine copy of the $y_-$–coordinate plane (or $y_+$–coordinate plane) based at the fixed point $(x, 0) \in \mathcal{F}_0(x)$ limit down to that fixed point in forward (or backward) time. These are the only subsets with those properties: any initial condition lying outside these subspaces yields an unbounded solution both in forward and backward time, so that we can write

$$\begin{align*}
\mathcal{W}_0^-(x, 0) &= \{(x, y) \mid \lim_{\tau \to -\infty} \phi^0_\ell(x, y) = x\} = \{x\} \times \mathbb{R}^\ell - \{0\}^\ell_- \subset \mathcal{F}_0(x), \\
\mathcal{W}_0^+(x, 0) &= \{(x, y) \mid \lim_{\tau \to +\infty} \phi^0_\ell(x, y) = x\} = \{x\} \times \{0\}^\ell_+ \times \mathbb{R}^\ell_+ \subset \mathcal{F}_0(x). 
\end{align*}$$

(3.5)

These subspaces are termed the stable (–) and unstable (+) subspaces of the fixed point $(x, 0)$, and they organize the dynamics on the fast fiber $\mathcal{F}_0(x)$. This is abundantly evident in (3.4), which describes the dynamics of an arbitrary initial condition and only involves, one, the projection of that initial condition on those subspaces; and two, the generators of the dynamics, $\Lambda_\pm^0$, on them.

Note, finally, that, for any fixed $x$, the $\ell$–dimensional fiber $\mathcal{F}_0(x)$ is foliated by the $\ell_-$–dimensional family 

$$\{z + \mathcal{W}_0^-(x, 0) \mid z \in \mathcal{W}_0^-(x, 0)\}$$

of $\ell_-$–dimensional manifolds. In plain words, this family consists of copies of the stable manifold (i.e., of the $\ell_-$–coordinate plane), cf. (3.5).

The family has been parameterized by the basepoints $z$

\footnote{Recall that $\sigma(\Lambda^0_+) \cup \sigma(\Lambda^0_-) = \sigma(\Lambda^0)$ and that $0 \not\in \sigma(\Lambda^0)$, since $\sigma(\Lambda^0) \cap i\mathbb{R} = \emptyset$ by assumption. Hence, $\Lambda^0_\pm$ have no zero eigenvalues and can thus be inverted.}

\footnote{Recall the form assumed by the exponential of a Jordan block (without forgetting to account for secular terms).}
of these copies located on the unstable manifold. None of these copies is independently invariant, except for the one going through the fixed point at the origin. The family is, nevertheless, invariant as a whole in the sense that each of its members is mapped to another member by the flow,

$$\phi_{\tau}^0(z + W^{0}(x,0)) = \phi_{\tau}^0(z) + W^0(x,0), \quad \text{for all } \tau.$$ 

In that sense, the evolution of each such copy (fiber) is dictated by the evolution of its basepoint on the unstable manifold. Identical, mutatis mutandis, results hold for the $\ell_-$-dimensional family $\{z + W^{0}_+(x,0)\}_{z \in \mathcal{W}^0(x,0)}$ of $\ell_+$-dimensional manifolds. The invariance condition for these two families, together with the observation that a member of each family goes through any given point in $z \in \mathcal{F}^0(x)$, makes precise the statement that the dynamics are organized by the stable/unstable manifolds. Indeed, to flow $z = (x, y_-, y_+)$ forward, one only has to obtain its basepoints $z_- = (z, y_-, 0) \in \mathcal{W}^0(x,0)$ and $z_+ = (z, 0, y_+) \in \mathcal{W}^0_+(x,0)$ by projecting $z$ on each manifold along the other (here: along the coordinate planes). Then,

$$\phi_{\tau}^0(z) = \left(\phi_{\tau}^0(z_-) + \mathcal{W}^0_+(x,0)\right) \cap \left(\phi_{\tau}^0(z_+) + \mathcal{W}^0(x,0)\right). \quad (3.6)$$

Interpreting these conclusions in terms of the original $w$-coordinates is a straightforward exercise. The subspace of fixed points, $\mathcal{M}^0$, corresponds to $\mathcal{U}^0$. Associated with any point $p \in \mathcal{U}^0$ is a fast fiber $\mathcal{F}^0(p)$, which is an affine copy of $\mathcal{V}^0$; namely, $\mathcal{F}^0(p) = p + \mathcal{V}^0 := \{p + v \mid v \in \mathcal{V}^0\}$. The dynamics on $\mathcal{F}^0(p)$ are organized around the stable and unstable (affine) subspaces of $p$; these are defined as the $\omega$– and $\alpha$–limit sets of that point and are affine copies of $\mathcal{V}^0_\pm$ in particular, $\mathcal{W}^0_\pm(p) = p + \mathcal{V}^0_\pm$. The phase space is foliated by the family of fast fibers based on the subspace of fixed points, $\mathbb{R}^{n+\ell} = \bigcup_{p \in \mathcal{U}^0} \mathcal{F}^0(p)$, so that any initial condition evolves on a specific fiber according to the hyperbolic dynamics associated with $(\mathcal{V}^0_+, \Lambda^0_+)$ and $(\mathcal{V}^0_-, \Lambda^0_-)$. Additionally, each fiber $\mathcal{F}^0(p)$ is itself foliated by affine copies of the stable and unstable manifolds,

$$\mathcal{F}^0(p) = \bigcup_{q \in \mathcal{V}^0} \left(q + \mathcal{V}^0_\pm\right) = \bigcup_{q \in \mathcal{V}^0} \left(q + \mathcal{V}^0_\pm\right), \quad (3.7)$$

with each such collection forming an invariant family:

$$\phi_{\tau}^0(q + \mathcal{V}^0) = \phi_{\tau}^0(q) + \mathcal{V}^0_\pm \quad \text{and} \quad \phi_{\tau}^0(q + \mathcal{V}^0) = \phi_{\tau}^0(q) + \mathcal{V}^0_\pm, \quad \text{for all } \tau. \quad (3.8)$$

The flow in $\mathbb{R}^N$ is obtained by that on the stable and unstable eigenspaces through the counterpart of (3.6),

$$\phi_{\tau}^0(q) = \left(\phi_{\tau}^0(q_-) + \mathcal{V}^0_+\right) \cap \left(\phi_{\tau}^0(q_+) + \mathcal{V}^0\right).$$

### 3.1.3 The $\varepsilon > 0$ dynamics

We now investigate how much of the structure presented above for the $\varepsilon = 0$ dynamics carries over to the far more interesting $\varepsilon > 0$ scenario.

We begin by briefly characterizing $\sigma(A^\varepsilon)$\footnote{The interested reader can find much more information in [Stewart & Sun, Ch. IV].}. The eigenvalues of $A^0$ perturb as eigenvalues of $A^\varepsilon$, under the assumptions for $A^0$ laid out in the previous section and as long as $A^\varepsilon = A^0 + \varepsilon R^\varepsilon$ with $R^\varepsilon$ a bounded perturbation. In concrete terms,

$$\sigma(A^\varepsilon) = \{\varepsilon \mu_j^\varepsilon\}_{1 \leq j \leq k} \cup \{\lambda_j^\varepsilon\}_{1 \leq j' \leq m'},$$

where $k \leq n$ and $m' \leq \ell$ are numbers\footnote{In general, $m' \geq m$. This is so because an eigenvalue with algebraic multiplicity greater than one may branch off into distinct eigenvalues, if its geometric multiplicity is strictly smaller than its algebraic one. Distinct eigenvalues, on the other hand, will remain distinct for small enough $\varepsilon$.} the coefficients $\mu^\varepsilon_1, \ldots, \mu^\varepsilon_k$ are $\mathcal{O}(1)$ (i.e., bounded) as $\varepsilon \downarrow 0$ and, for every $j' < m'$, there exists $j < m$ such that $\lambda_j^\varepsilon \rightarrow \lambda_j^0$ as $\varepsilon \downarrow 0$. For all small enough values of $\varepsilon$, these two sets of eigenvalues can be kept disjoint. At the same time, the $\varepsilon = 0$ eigenspaces also perturb to $\mathcal{U}^\varepsilon$, $\mathcal{V}^\varepsilon_\pm$ with dimensions equal to their $\varepsilon = 0$ counterparts. In particular, although the number of distinct eigenvalues with positive (or negative) real parts may change due to branchings, the numbers $n$ and $\ell_\pm$ are $\varepsilon$–independent, for
\(\varepsilon\) within a neighborhood of zero. The reason for that is, essentially, that dimension is a discrete entity whereas these subspaces change continuously—\(i.e.,\) limit to their \(\varepsilon = 0\) counterparts as \(\varepsilon \downarrow 0\).7

Working as in the previous section, we can introduce a matrix \(P^\varepsilon = [U^\varepsilon, V^\varepsilon_-, V^\varepsilon_+]\), whose columns span the three invariant eigenspaces introduced above, and use its inverse to define new coordinates \(z = (P^\varepsilon)^{-1} w\). Choosing the columns of \(U^\varepsilon\) to be generalized eigenvectors of \(A^\varepsilon\), we arrive at the system

\[
\begin{align*}
x' &= \varepsilon M^\varepsilon x, \\
y_- &= \Lambda^\varepsilon_- y_-, \\
y_+ &= \Lambda^\varepsilon_+ y_+.
\end{align*}
\]

This is the \(\varepsilon > 0\) analog of \((3.3)\), with \(\varepsilon M^\varepsilon = V^\varepsilon_\ell A^\varepsilon U^\varepsilon\) the Jordan block associated with the small, \(O(\varepsilon)\) eigenvalues. The flow generated by this linear system is

\[
\phi^\varepsilon(x, y) = \begin{pmatrix} e^{\varepsilon \tau M^\varepsilon x} \\ e^{\tau \Lambda^\varepsilon_-} y_- \\ e^{\tau \Lambda^\varepsilon_+} y_+ \end{pmatrix};
\]

this is the \(\varepsilon > 0\), direct counterpart of \((3.4)\). These dynamics are also easy to fathom and can be contrasted directly to those of \((3.3)\); we do this directly in terms of the original \(w\)-coordinates. First, the subspace \(M^0 = U^0\) of fixed points perturbs to a nearby subspace \(M^\varepsilon = U^\varepsilon\). The dynamics on that subspace are not trivial (stationary) but, rather, dictated by the \(x\)-component of the flow \(\phi^\varepsilon\). Next, \(M^\varepsilon\) is equipped with a foliation of fast fibers, \({\mathcal{F}}^\varepsilon(p)\}_{p \in M^\varepsilon}\), with each fiber being an affine copy of the complementary eigenspace: \(\mathcal{F}^\varepsilon(p) = p + V^\varepsilon\). The base point \(p \in M^\varepsilon\) of each such fiber evolves in the way outlined above. As a result, the fiber \(\mathcal{F}^\varepsilon\) is, in general, no longer invariant. The flow does, nevertheless, map each fiber to another one through the rule

\[
\phi^\varepsilon(\mathcal{F}^\varepsilon(p)) = \mathcal{F}^\varepsilon(\phi^\varepsilon(p));
\]

as the reader can easily (and must) show. In words, the fiber based at \(p = M^\varepsilon\), at time zero, is mapped after time \(\tau\) to the fiber based at \(\phi^\varepsilon(p) \in M^\varepsilon\); the slow dynamics on \(M^\varepsilon\) entirely determines the evolution of the entire fiber by acting on its basepoint. The fast dynamics transversal to \(M^\varepsilon\)—that is, the precise point on \(\mathcal{F}^\varepsilon(\phi^\varepsilon(p))\) that \(q \in \mathcal{F}^\varepsilon(p)\) is mapped to—are determined from the evolution of the \(y\)-components in \((3.10)\).

Finally, note that each individual fiber \(\mathcal{F}^\varepsilon(p)\) is itself foliated by affine copies of the eigenspace \(V^\varepsilon_-\) (or \(V^\varepsilon_+\)), with each copy based on \(p + V^\varepsilon_-\) (or on \(p + V^\varepsilon_+)\). These lower-dimensional subfibers are not individual either, but their union over the entire family \({\mathcal{F}}^\varepsilon(p)\}_{p \in M^\varepsilon}\) is: each member is mapped to another member by the flow, much like our \(\varepsilon = 0\) result \((3.8)\).

Perhaps the most notable difference between the \(\varepsilon > 0\) and \(\varepsilon = 0\) cases concerns the loss of invariance of each individual fiber. Although no isolated, invariant, \(\varepsilon = 0\) fiber can be said to perturb as an independent entity—it is not clear which member of \({\mathcal{F}}^\varepsilon(p)\}_{p \in M^\varepsilon}\) corresponds to a given \(\mathcal{F}^\varepsilon(p')\), since \(p' \notin M^\varepsilon\) in general—the family \({\mathcal{F}}^\varepsilon(p)\}_{p \in M^\varepsilon}\) of fast fibers perturbs collectively as an invariant family. An immediate consequence of this loss of invariance is that the definitions \((3.5)\) now become inapplicable. In particular, one cannot demand that the basepoint \(p\) of a fiber \(\mathcal{F}^\varepsilon(p)\) is the \(\alpha-\) or \(\omega-\)limit set of any point on that fiber, since \(p\) itself is not invariant. We can, nevertheless, define the affine subspaces \(W^\varepsilon_{\pm}(p) = p + V^\varepsilon_{\pm}\) and observe that they have the following property:

\[
\forall q \in W^\varepsilon_{\pm}(p) \lim_{\tau \to \pm \infty} ||\phi^\varepsilon(\tau)(q) - \phi^\varepsilon(p)|| = 0.
\]

In other words, all points on the stable/unstable manifold of a point \(p \in M^\varepsilon\) approach in forward/backward time the forward/backward flow of that point. Note that this property does not suffice to recover the relation \(W^\varepsilon_{\pm}(p) = p + V^\varepsilon_{\pm}\). If, for example, the origin is a globally attracting fixed point for the system \(u^\varepsilon = A^\varepsilon u\), then—irrespective of \(p\)—all points in \(\mathbb{R}^{n+\varepsilon}\) have the forward-contraction property since they limit—together with \(p\) itself—to the origin in forward time.

7The notion of a subspace limiting to another one requires equipping the set of subspaces with a specific topology. This can be, for example, the metric topology associated with the Grassmannian metric, but we will refrain from elaborating further.
3.1.4 A rescale of time

We conclude our discussion of the linear problem \( w' = A^\varepsilon w \) by looking at another distinguished limit. We start by rescaling time via \( t = \varepsilon \tau \), which corresponds to a compression of time by a factor of \( \varepsilon \): an \( O(1) \) interval of time \( \tau \) corresponds to an \( O(\varepsilon) \) such interval in terms of \( t \). Using this rescaling and writing \( (\cdot)' = d \cdot /dt \), we can recast (3.9) in the form

\[
\begin{align*}
\dot{x} &= M^\varepsilon x, \\
\varepsilon \dot{y}_- &= \Lambda^\varepsilon_- y_-, \\
\varepsilon \dot{y}_+ &= \Lambda^\varepsilon_+ y_+;
\end{align*}
\]

(3.13)

this corresponds to the ODE system \( \varepsilon \dot{w} = A^\varepsilon w \). We denote the flow generated by this system by \( \varphi^\varepsilon_t \). Naturally, for \( \varepsilon > 0 \), (3.9) and (3.13) produce flows related through \( \varphi^\varepsilon_t = \varphi^0_1/t\varepsilon \). In phase space terms, the two flows produce identical trajectories: the rescaling of time merely yields a reparameterization of those trajectories.

The situation is markedly different for \( \varepsilon < 0 \). Using that the matrices \( \Lambda^0_{\pm} \) are invertible, we can rewrite this system as

\[
\begin{align*}
\dot{x} &= M^0 x, \\
0 &= \Lambda^0_- y_-, \\
0 &= \Lambda^0_+ y_+.
\end{align*}
\]

(3.14)

Using that the matrices \( \Lambda^0_{\pm} \) are invertible, we can rewrite this system as

\[
\dot{x} = M^0 x \quad \text{and} \quad y = 0.
\]

Clearly, this is a system of differential–algebraic equations (DAEs) composed of an algebraic constraint \( (y = 0) \) and an ODE \( (\dot{x} = M^0 x) \). The algebraic equation constrains the evolution on the \( \varepsilon = 0 \) slow subspace \( M^0 \), while the ODE equips this subspace with nontrivial dynamics. This situation is in stark contrast with our interpretation of the dynamics associated with (3.3), which included nontrivial dynamics outside \( M^0 \) and trivial (stationary) dynamics on it. Given the picture of the \( \varepsilon > 0 \) dynamics which we painted above, this nontrivial flow on \( M^0 \) can be geometrically interpreted as the \( \varepsilon \downarrow 0 \) limit of the dynamics on \( M^\varepsilon \) (recall that \( M^\varepsilon \rightarrow M^0 \), as \( \varepsilon \downarrow 0 \)). From the reverse point of view, the leading order, \( \varepsilon > 0 \) dynamics can be seen as the concatenation of, first, the fast, hyperbolic dynamics along the fast fibers—associated with (3.3)—and, second, the slow, drifting dynamics on \( M^0 \)—associated with (3.14).

In either case, the fact that \( M^0 \) appears fixed in the \( \varepsilon = 0 \) fast formulation makes sense as long as we remember that \( \varepsilon \) measures the timescale disparity between fast (on \( F^\varepsilon \)) and slow (on \( F^0 \)) dynamics. In the \( \tau \)–timescale, the dynamics along the fast fibers evolve in an \( O(1) \) timescale, whereas those on the slow manifold in a much longer \( O(1/\varepsilon) \) timescale. Hence, these slow dynamics appear entirely stationary, when \( \varepsilon \) is reduced to zero. Similarly, in the \( t \)–timescale, it is the slow dynamics that evolve in an \( O(1) \) timescale, with their fast counterparts evolving on an \( O(\varepsilon) \) timescale. Hence, these fast dynamics are instantaneously exhausted and the evolution constrained on \( M^0 \) already from the start.

3.2 Setting of the nonlinear problem

In this section, we concern ourselves with ODE systems of the form

\[
\begin{align*}
x' &= \varepsilon f(x, y; \varepsilon), \\
y' &= g(x, y; \varepsilon).
\end{align*}
\]

(3.15)

Here, the following quantities are at play:

- The independent variable \( \tau \in \mathbb{R}_+ = [0, \infty) \), supposedly measuring time.
- The dependent variables \( x : \mathbb{R}_+ \rightarrow \mathbb{R}^n \) and \( y : \mathbb{R}_+ \rightarrow \mathbb{R}^\ell \), which are multidimensional functions of time of dimension \( n \) and \( \ell \), respectively. For the purposes of these notes, we will be assuming that the state vector \((x, y)\) is constrained to lie within an open set \( U \subseteq \mathbb{R}^{n+\ell} \). Typically, \( U \) will be forward invariant, that is, initial conditions \((x, y) \in U\) will generate trajectories \( \{ \phi^\varepsilon_\tau (x, y) \mid \tau \in \mathbb{R}_+ \} \subset U \).
- The parameter \( \varepsilon \in I = [0, \bar{\varepsilon}] \), which is a small parameter \( (\varepsilon \ll 1) \) measuring the timescale disparity between the dynamics of \( x \) and those of \( y \).
• And finally, the functions $f : U \times I \to \mathbb{R}^n$ and $g : U \times I \to \mathbb{R}^\ell$, which are smooth functions of their arguments (including in $\varepsilon$). Technically speaking, $f, g \in C^\infty(U \times I)$.

Note that this explicit fast–slow form is not the most general one can consider, since it assumes that the state variables are already partitioned in fast ($y$) and slow ($x$) classes. This, for example, was not the case for our linear system $w' = A^\varepsilon w$, where all state variables are generically fast.$^8$ It is clearly the case for the transformed system (3.13)—for which $f$ and $g$ do not even depend on $y$ and $x$, respectively—and it would also be the case had we defined new coordinates $\bar{z} = (P^0)^{-1} w$ (compare to $z = (P^\varepsilon)^{-1} w$).

Here also, we can plainly rescale time by $t = \varepsilon \tau$ to put (3.15) into the form

$$\begin{align*}
\dot{x} &= f(x, y; \varepsilon), \\
\varepsilon \dot{y} &= g(x, y; \varepsilon),
\end{align*}$$

(3.16)

with $\dot{\cdot} = d \cdot /dt$. Clearly, (3.15) and (3.16) are equivalent as long as $\varepsilon > 0$, in the sense that they produce the same phase portrait. The sole effect of switching from $\tau$ to $t$ is a reparameterization of the phase space trajectories, that is, a rescaling of the speed with which these trajectories are traversed. In particular, trajectories are traversed faster (by a factor of $1/\varepsilon$) in the formulation (3.16)—the so-called slow formulation—compared to (3.15)—the so-called fast formulation. We will see imminently that this equivalence between the two phase portraits breaks down when $\varepsilon = 0$, just like it did for our linear system.

### 3.3 The $\varepsilon = 0$ dynamics

The fast and slow systems, (3.15) and (3.16) respectively, have two distinct distinguished limits for $\varepsilon = 0$:

$$z' = \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 \\ g_0(z) \end{pmatrix} =: G_0(z) \quad \text{and} \quad \begin{pmatrix} \dot{x} \\ \varepsilon \dot{y} \end{pmatrix} = \begin{pmatrix} f_0(z) \\ g_0(z) \end{pmatrix}, \quad \text{with} \quad \begin{cases} f_0(z) := f(x, y; 0), \\ g_0(z) := g(x, y; 0). \end{cases}$$

(3.17)

The $\varepsilon = 0$ limit of the fast system (left) exhibits a specific set of dynamical characteristics, which we summarize below and demonstrate in Fig. [coming up]. The reader should make the correspondence between these and analogous entities introduced in our linear context of Section 3.1.

• For each fixed value of $x \in \mathbb{R}^n$, the $\ell$–dimensional, affine linear space $\mathcal{F}^0(x) = \{(x, y) | y \in \mathbb{R}^\ell \}$ is invariant; initial conditions on $\mathcal{F}^0(x)$ remain on $\mathcal{F}^0(x)$ for all times. That is so because $x' = 0$, and each $\mathcal{F}^0(x)$ is called an (unperturbed) fast fiber.

• The fixed points of the system are determined by $g_0(x, y) = 0$. Since these are $\ell$ algebraic equations in $n + \ell$ unknowns, they generically (but not automatically) define an $n$–dimensional set of solutions $\mathcal{M}^0 = \{(x, y) | g_0(x, y) = 0 \}$ called the (unperturbed) slow manifold for our system. Below, we will be wanting to formulate conditions ensuring that $\mathcal{M}^0$ is an $n$–dimensional manifold embedded in $\mathbb{R}^{n + \ell}$.

• For every $x \in \mathbb{R}^n$ for which $g_0(x, y) = 0$ has at least one solution $y$, the point $(x, y) \in \mathcal{M}^0 \cap \mathcal{F}^0(x)$ is a fixed point of the flow on $\mathcal{F}^0(x)$. Below, we will be wanting to characterize the transversal dynamics around that point; in particular, we will be assuming that these dynamics are hyperbolic when constrained on the the invariant set $\mathcal{F}^0(x)$.

The $\varepsilon = 0$ limit of the slow system (right), on the other hand, is a system of differential–algebraic equations (DAEs) exhibiting entirely different characteristics (see, here also, Fig. [coming up]).

• The flow is defined at most for points $(x, y)$ satisfying the algebraic condition $g_0(x, y) = 0$. In other words, the flow is certainly not defined outside the manifold $\mathcal{M}^0$.

• As is evident from (3.17), the DAE in question only dictates the evolution of the $x$–variables. That of the $y$–variables is dictated by the demand that $\mathcal{M}^0$ be invariant, that is, that the algebraic equation $g_0(x, y) = 0$ be satisfied at all times.

$^8$To see this, work your way from (3.13) backwards to (3.1) by an application of the transform $z \mapsto w$. 
The pictures sketched by these two distinguished limits appear complementary at best. The fast system yields a foliation of the phase space into an $\ell$–dimensional family of $n$–dimensional fast fibers. Each fast fiber is invariant and contains nonlinear, $n$–dimensional dynamics. Additionally, each such fiber contains at least one fixed point—assumed hyperbolic—and the union of these fixed points yields an $n$–dimensional manifold $M^0$. Plainly, the dynamics on $M^0$ are trivial: all of its points are fixed under the flow. The slow system, on the other hand, paints a picture where dynamics—typically nonlinear—only exist on $M^0$. Naively speaking, one’s expectation is that the two pictures can be combined in one, as the case was for our linear problem. Below, we work out the conditions under which this is true and explain in which sense this combination is valid. Perhaps unsurprisingly, they all turn out to concern the local behavior of the system near $M^0$.

### 3.4 A fundamental hypothesis: normal hyperbolicity

To motivate our discussion below, we consider an arbitrary point $z = (x, y) \in M^0$ and the linearized dynamics around it in the $\varepsilon = 0$, fast formulation:

$$
\begin{pmatrix}
\delta x \\
\delta y 
\end{pmatrix} = 
\begin{pmatrix}
D_x g_0(z) & 0 \\
0 & D_y g_0(z)
\end{pmatrix}
\begin{pmatrix}
\delta x \\
\delta y
\end{pmatrix}
$$

or, compactly, \( \dot{z} = D G_0(z) \delta z \).

The eigenvalues of the Jacobian matrix $D G_0(z)$ are rather easy to characterize by means of the following:

**Statement #0.** The spectrum of the Jacobian is given by

$$
\sigma(D G_0(z)) = \{0\} \cup \sigma(D_y g_0(z)).
$$

The zero eigenvalue is of algebraic and geometric multiplicities no smaller than $n$. □

This statement leads to the following fundamental hypothesis:

**Normal Hyperbolicity Assumption.** For every $z \in M^0$,

$$
0 \notin \sigma(D_y g_0(z)) \quad \text{or, equivalently,} \quad \det(D_y g_0(z)) \neq 0.
$$

Under this assumption, we can refine the above statement to read:

**Statement #1.** The spectrum of the Jacobian is given by

$$
\sigma(D G_0(z)) = \{0\} \cup \sigma(D_y g_0(z)).
$$

Both the algebraic and geometric multiplicities of the zero eigenvalue are equal to $n$. The corresponding invariant subspace is $T_z M^0$—the space tangent to $M^0$ at $z$. The eigenvalues contained in $\sigma(D_y g_0(z))$ are all associated with the the $\ell$–dimensional subspace $\{0\} \times \mathbb{R}^\ell$, which $D G_0(z)$ leaves invariant. □

You are asked to prove this statement in Homework #03. For the time being, note that this statement is the precise analog of the statement, in our linear problem, that $\text{Ker}(A^0)$ is $n$–dimensional and that the eigenvalues of the mapping induced by $A^0$ on $\mathbb{R}^{n+\ell}/\text{Ker}(A^0)$ are bounded away from the imaginary axis. Note, additionally, that one could have guessed the result. Plainly, the zero eigenvalue is due to all points on $M^0$ being fixed: this information is picked up by the linearized dynamics, which dictates that perturbations ‘along $M^0$’ (i.e., in the tangent directions) stay put (i.e., do not evolve in time). Similarly, the remaining eigenvalues are associated with the fast fiber $\mathcal{F}^0(z)$ which remains invariant under the $\varepsilon = 0$ fast flow; indeed, $\mathcal{F}^0(z)$ is but an affine copy of the invariant subspace $\{0\} \times \mathbb{R}^\ell$ based at $z$. Just as $g_0(x, \cdot)$ governs the full, nonlinear dynamics on that fiber, so does $D_y g_0(z)$ govern the linearized dynamics on the same fiber around the fixed point $z$.

The assumed property $\det(D_y g_0(z)) \neq 0$ is called normal hyperbolicity, as it is equivalent to the demand that $z \in M^0$ is a hyperbolic fixed point for $G_0|_{\mathcal{F}^0(z)}$—the restriction of $G_0$ on $\mathcal{F}^0(z)$.$^9$ An interesting and related corollary to that condition is the following:

---

$^9$This is purposefully vague. In reality, each fiber might contain more than one fixed points, in which case one will be forced to choose which points to glue together to obtain a manifold.

$^10$Recall the definition of hyperbolicity: given a continuously differentiable vector field $v : \mathbb{R}^k \to \mathbb{R}^k$ (for some $k$) and a point $z \in \mathbb{R}^k$ which remains fixed under $v$ (i.e., $v(z) = 0$), we will be calling $z$ hyperbolic if all eigenvalues of the Jacobian $D v(z)$ have nonzero real parts (i.e., $\sigma(D v(z)) \cap \mathbb{R} = \emptyset$).
Statement #2. Let \( x \in \mathbb{R}^n \) be fixed and assume that \( z = (x, y) \in \mathcal{M}^0 \). Then, the point \( z \) is an isolated solution of \( g_0(x, y) = 0 \). Additionally, for small enough \( \varepsilon \), there exists a point \( (x, y(\varepsilon)) \) satisfying \( g(x, y(\varepsilon); \varepsilon) = 0 \) and \( y(\varepsilon) \to y \) as \( \varepsilon \downarrow 0 \). Finally, the \( \varepsilon = 0 \) slow manifold \( \mathcal{M}^0 \) and the \( \varepsilon = 0 \) slow fiber \( \mathcal{F}^0(x) \) intersect transversally: \( T_x \mathcal{M}^0 \oplus T_x \mathcal{F}^0(z) = \mathbb{R}^{n+\ell} \). \( \square \)

The first part of the above statement is a direct consequence of the Implicit Function Theorem, the main condition for the applicability of which is precisely what we called normal hyperbolicity. The second part is the precise analog of the statement \( \mathbb{R}^{n+\ell} = \mathcal{U}^0 \oplus \mathcal{V}^0 \) for our linear problem; you are called to prove it in Homework #03.

Connectivity and Compactness Assumptions. Before we proceed, we assume further that \( \mathcal{M}^0 \) is a connected and compact manifold. Statement #2 dictates, then, that there exists a compact set \( K \subset \mathbb{R}^n \) and a unique function \( h_0: K \to \mathbb{R}^\ell \) such that

\[
\mathcal{M}^0 = \text{graph}(h_0) = \{(x, h_0(x)) \mid x \in K\}.
\]

The final corollary of our normal hyperbolicity assumption concerns the evolution, on \( \mathcal{M}^0 \) and for the \( \varepsilon = 0 \) slow system, of the \( y \)-variables.

Statement #3. The evolution, in the \( \varepsilon = 0 \) slow formulation, on the manifold \( \mathcal{M}^0 \) is dictated by the ODE system

\[
\begin{align*}
\dot{x} &= f_0(x, y), \\
\dot{y} &= -(D_y g_0(x, y))^{-1} D_x g_0(x, y) f_0(x, y).
\end{align*}
\]

This can be proven by writing \( \varphi^0(x, y) \) for the flow on \( \mathcal{M}^0 \), substituting into the \( \varepsilon = 0 \) DAE \((3.17)\), taking the total time derivative of (both sides of) the algebraic constraint, and solving the resulting equation for \( y \). Note that the role of normal hyperbolicity is to ensure that \( (D_y g_0)^{-1} \) exists. An alternative way of describing evolution on \( \mathcal{M}^0 \) remains, of course, \((3.17)\) with the algebraic constraint solved for \( y \),

\[
\dot{x} = f_0(x, h_0(x)) \quad \text{and} \quad y = h_0(x).
\]

3.5 Straightening it all out

Before elucidating further the \( \varepsilon = 0 \), fast dynamics in an \( \mathcal{O}(1) \) neighborhood of \( \mathcal{M}^0 \), we change coordinates in \((3.17)\) from \((x, y)\) to a set which is dynamically more meaningful. We undertake this task in the form of a roadmap to facilitate reading.

Straightening out \( \mathcal{M}^0 \). Our first step is to coordinatize \( K \times \mathbb{R}^\ell \) via

\[
\hat{z} = (\hat{x}, \hat{y}) = (x, y - h_0(x)).
\]

The introduction of these new coordinates flattens out the unperturbed slow manifold, in the sense that this becomes \( \mathcal{M}^0 = \{(x, 0) \mid x \in K\} = K \times \{0\}^\ell \); it lies fully within the \( \hat{x} \)-coordinate plane. The unperturbed fast fibers, on the other hand, remain of the form \( \hat{x} = \text{const} \).

The ODEs dictating the evolution in these new coordinates are obtained by differentiating both sides of \((3.19)\) and employing the corresponding ODEs \((3.17)\) for the fast system. We find

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix}' =
\begin{pmatrix}
0 & 0 \\
( g_0(\hat{x}, \hat{y} + h_0(\hat{x})) & 0
\end{pmatrix} =: G_0(\dot{z}).
\]

Note that \( \mathcal{M}^0 \) retains its normally hyperbolic character under this change of coordinates, since

\[
\sigma \left( DG_0(x, 0) \right) = \sigma \left( DG_0(x, h_0(x)) \right) \quad \text{by virtue of} \quad DG_0 = \begin{pmatrix}
0 & 0 \\
D_x g_0 + D_y g_0 D h_0 & D_y g_0
\end{pmatrix} = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix} \quad \text{[11]}
\]

\[11\] Here, we have used that \( D_x g_0 + D_y g_0 D h_0 = 0 \) on \( \mathcal{M}^0 \), which is obtained by differentiating the identity \( g_0(x, h_0(x)) = 0 \).
Note, also, that the Jacobian is block-diagonal in these new coordinates, as one would expect from knowing that $\mathcal{M}^0 \subset \mathbb{R}^n \times \{0\}^\ell$ and $\mathcal{F}^0(x, 0) = (x, 0) + \{0\}^n \times \mathbb{R}^\ell$. Below, we drop the hats in all quantities with a slight abuse of notation.

**Stable and unstable manifolds.** Before we proceed further, we define the stable and unstable manifolds of each point $z \in \mathcal{M}^0$ and then of the entire slow manifold $\mathcal{M}^0$.

Given $z \in \mathcal{M}^0$, the linearized dynamics around it and on $\mathcal{F}^0(z)$ are governed by $D_y \mathcal{F}^0_0(z)$, the spectrum of which only contains hyperbolic (i.e., non-imaginary) eigenvalues. In general, then, $z$ is a hyperbolic saddle point under the (nonlinear) dynamics generated by $\mathcal{F}_0(z)$, with $\ell_-$ stable and $\ell_+$ unstable eigendirections. We denote by $W^0_\pm(z) \subset \mathcal{F}^0(z)$ the (global) stable and unstable manifolds of $z$,

\[
W^0_-(z) = \left\{ z_0 \in \mathcal{F}^0(z) \left| \lim_{t \to \infty} \phi^0_t(z_0) \to z \right. \right\} \quad \text{and} \quad W^0_+(z) = \left\{ z_0 \in \mathcal{F}^0(z) \left| \lim_{t \to -\infty} \phi^0_t(z_0) \to z \right. \right\},
\]

where $\{\phi^0_t\}_{t \geq 0}$ denotes the flow generated by the full, nonlinear vector field.

Note that $z \in W^0_-(z) \cap W^0_+(z)$, since $z$ is fixed under $\phi^0_t$. Note, also, that the dimensions $\ell_\pm$ are necessarily independent of $z \in \mathcal{M}^0$. Hence, the stable/unstable manifolds are of the same dimension for all $z \in \mathcal{M}^0$.

Using this information, we can define the $(n + \ell_-)$-dimensional unstable/stable manifolds of $\mathcal{M}^0$,

\[
W^0_\pm(\mathcal{M}^0) = \bigcup_z W^0_\pm(z), \quad \text{whence also} \quad \mathcal{M}^0 \subset W^0_-(\mathcal{M}^0) \cap W^0_+(\mathcal{M}^0).
\]

Finally, we glue (tangent) stable and unstable directions to define another pair of manifolds. Specifically, for each $z \in \mathcal{M}^0$, consider the affine subspaces $z + T_z W^0_\pm(z)$; these are tangent to $W^0_\pm(z)$ at the point $z$, by the stable/unstable manifold theorem, and vary smoothly with $z$. GLueing these subspaces together, we obtain two other manifolds,

\[
TW^0_\pm(\mathcal{M}^0) = \bigcup_{z \in \mathcal{M}^0} \left( z + T_z W^0_\pm(z) \right).
\]

At the next step in our roadmap, we will devise a coordinate change to straighten these manifolds out.

**Straightening out $TW^0_\pm(\mathcal{M}^0)$.** Let $z = (x, 0) \in \mathcal{M}^0$. Although the $y = 0$ fast fiber $\mathcal{F}^0(z)$ is itself straightened out\footnote{The proof is by contradiction: indeed, $\ell_\pm$ corresponds to the number of stable or unstable eigenvalues. If these numbers were to change as $z$ varied, then at least one eigenvalue would have to cross from left to right half-plane or vice versa. Seeing as each eigenvalue depends continuously on $z$, as the eigenvalue in question would need to cross through the imaginary axis and as $\mathcal{M}^0$ was assumed compact, we conclude that said eigenvalue would become imaginary at a specific value $z \in \mathcal{M}^0$. This is a clear violation of normal hyperbolicity, whence the contradiction.}, this is not the case with its submanifolds $W^0_\pm(z) \subset \mathcal{F}^0(z)$. In this step, we change coordinates to make the tangent spaces to these two manifolds (i.e., the stable/unstable directions) orthogonal to each other and to $\mathcal{M}^0$; our work here is heavily indebted to the linear case considered earlier.

Since $z$ is hyperbolic under the dynamics on $\mathcal{F}^0(z)$, we have $T_z \mathcal{F}^0(z) = T_z W^0_-(z) \oplus T_z W^0_+(z)$. Let, now, \{\hat{v}_1^0(z), \ldots, \hat{v}_\ell^0(z)\} be a basis for $T_z \mathcal{F}^0(z)$ which respects this partition:

\[
T_z W^0_-(z) = \text{span} \{\hat{v}_1^0(z), \ldots, \hat{v}_{\ell_-}^0(z)\} \quad \text{and} \quad T_z W^0_+(z) = \text{span} \{\hat{v}_{\ell_-+1}^0(z), \ldots, \hat{v}_\ell^0(z)\}.
\]

Then, every point $z \in \mathbb{R}^N$ can be written as

\[
z = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} + \sum_{1 \leq i \leq \ell_-} \hat{y}_i \hat{v}_i^0(x, 0) + \sum_{1 \leq i \leq \ell_+} \hat{y}_{i+\ell_-} \hat{v}_{i+\ell_-}^0(x, 0),
\]

since

\[
\begin{pmatrix} 0 \\ y \end{pmatrix} \in T_z \mathcal{F}^0(z) = \{0\}^n \times \mathbb{R}^\ell.
\]

This decomposition defines a change of coordinates $z \mapsto \hat{z} = (\hat{x}, \hat{y})$, with $\hat{x} = x$ and $\hat{y} = (\hat{y}_-, \hat{y}_+)$ defined above. Plainly, $y = 0$ if and only if $\hat{y} = 0$, so $\mathcal{M}^0 = \{(\hat{x}, 0) \mid \hat{x} \in K\}$ in these coordinates as well. Similarly,\footnote{It is identified with an affine copy of the $y-$coordinate plane, $\mathcal{F}^0(z) = z + \{0\}^n \times \mathbb{R}^\ell$.}
\(F^0(\hat{z}) = \hat{z} + \{0\}^n \times \mathbb{R}^\ell\) here as well. Embedded in this fast fiber are, now, the two straightened out submanifolds

\[TW^0_0(\mathcal{M}^0) = K \times \mathbb{R}^\ell_- \times \{0\}^{\ell_+} \quad \text{and} \quad TW^0_1(\mathcal{M}^0) = K \times \{0\}^{\ell_-} \times \mathbb{R}^{\ell_+} \tag{3.21}\]

coinciding with the \(\hat{y}_-\) and \(\hat{y}_+\)-coordinate planes.

We conclude this step by deriving the governing ODEs in terms of the new coordinate system. Let \(e_j = (\delta_{ij})_{1 \leq i, j \leq N}\) be the \(j\)-th standard basis column vector (1 \(\leq j \leq n\)) and write

\[z = \begin{pmatrix} x \\ y \end{pmatrix} = P^0(x, 0) \hat{z}, \quad \text{with} \quad P^0(x, 0) = [e_1, \ldots, e_n, v^0_1(x, 0), \ldots, v^0_\ell(x, 0)] = [E, V^0_-(x, 0), V^0_+(x, 0)].\]

Then, we calculate

\[z' = P^0(x, 0) \hat{z}' + (P^0(x, 0))' \hat{z} = P^0(x, 0) \hat{z}', \quad \text{since} \quad (P^0(x, 0))' = D_P P^0(x, 0) x' = 0;\]

recall \(\hat{z}' = \hat{z} \hat{z} = (P^0(x, 0))^{-1} \hat{z}' = (P^0(x, 0))^{-1} G_0(z) = (P^0(x, 0))^{-1} G_0(z)(P^0(x, 0) \hat{z}) = : \hat{G}_0(\hat{z}).\)

The linearization of this vector field about any point \(\hat{z} \in \mathcal{M}^0\) may be calculated to be

\[D\hat{G}_0(x, \hat{y}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \tilde{A}_0^0(x) & 0 \\ 0 & 0 & \tilde{A}_0^0(x) \end{pmatrix}, \tag{3.22}\]

reflecting the invariance both of \(\mathcal{M}^0\) and of the coordinate subspaces \(\mathbb{R}^n \times \mathbb{R}^\ell_- \times \{0\}^{\ell_+}\) and \(\mathbb{R}^n \times \{0\}^{\ell_-} \times \mathbb{R}^{\ell_+}\) (cf. \(\mathcal{M}^0\)) under the linearized dynamics. Here, the spectrum of the \(\ell_+\) blocks \(\tilde{A}_0^0(x)\) matches the unstable/stable eigenvalues of \(Dg_0(x, 0)\). These blocks can naturally be put in Jordan canonical form by selecting \(\{v^0_1(z), \ldots, v^0_\ell(z)\}\) to be generalized eigenvectors of \(Dg_0(x, 0)\).

Here also, we shall drop hats in what follows.

**Straightening out \(W^0_\pm(\mathcal{M}^0)\).** We are now ready to introduce new coordinates to straighten out the nonlinear manifolds \(W^0_\pm(\mathcal{M}^0)\). In the vein of our work for \(TW^0_\pm(\mathcal{M}^0)\) above, we focus on straightening out \(W^0_\pm(z)\) for any specific \(z = (x, 0) \in \mathcal{M}^0\), since \(W^0_\pm(\mathcal{M}^0)\) are made by glueing such manifolds together.

To do this, we first note that the manifolds \(W^0_\pm(z)\) go through \(z\) and are tangent to \(\mathbb{R}^n \times \{0\}^{\ell_-} \times \mathbb{R}^{\ell_+}\) and \(\mathbb{R}^n \times \mathbb{R}^\ell_- \times \{0\}^{\ell_+}\), respectively; cf. \(\mathcal{M}^0\). By the stable/unstable manifold theorem, further, these manifolds are at least \(C^1\) and hence they are graphs over their tangent spaces in a neighborhood of \(z\). It follows that there exist open sets \(U_\pm(z) \subset \mathbb{R}^\ell\) containing \(z\) and functions \(\eta^0_\pm(\cdot; z) : U_\pm(z) \rightarrow \mathbb{R}^{\ell_\pm}\) such that

\[W^0_0(z) = \text{graph}(\eta^0_- (\cdot; z)) = \{(x, y, -\eta^0_+(y, x, y); y \in U_-(z))\}, \]
\[W^0_\pm(z) = \text{graph}(\eta^0_+ (\cdot; z)) = \{(x, \eta^0_+(y, x, y), y_+ \in U_+(z))\}.

The functions \(\eta^0_\pm(\cdot; z)\) satisfy, additionally,

\[\eta^0_\pm(0; z) = 0 \quad \text{and} \quad D\eta^0_\pm(0; z) = 0.\]

The first of these expresses that \(z \in W^0_\pm(z)\) and the second the aforementioned tangencies. The change of coordinates now suggests itself,

\[\hat{z} = (\hat{x}, \hat{y}_-, \hat{y}_+) = (x, y, -\eta^0_+(y_+, x, y, y_+), y_+ - \eta^0_-(y_-, x, y, y_+)),\]

compare to \(\mathcal{M}^0\). Under this—final—change of coordinates, we can straighten out the stable/unstable manifolds in a tubular neighborhood of \(\mathcal{M}^0\). Indeed, write \(U_\pm = \cap_{z \in \mathcal{M}^0} U_\pm(z)\) for a neighborhood of \(z \in \mathcal{M}^0\) over which \(\eta^0_\pm(\cdot; z)\) is defined. Then,

\[\mathcal{M}^0 = K \times \{0\}^{\ell_-} \times \{0\}^{\ell_+}, \quad W^0_0(\mathcal{M}^0) = \{0\}^n \times U_- \times \{0\}^{\ell_+} \quad \text{and} \quad W^0_1(\mathcal{M}^0) = \{0\}^n \times \{0\}^{\ell_-} \times U_+. \tag{3.23}\]

\(^{14}\text{Intuitively, } dP^0/\,dt = 0 \text{ follows from } P^0 \text{ being a (matrix) function on } \mathcal{M}^0 \text{ which, in turn, is fixed under the flow. Since points on } \mathcal{M}^0 \text{ are fixed, so are functions of these points.}\)
Fibration of a neighborhood of $\mathcal{M}^0$. We finally mention, without delving into lengthy details, that there exists an $\mathcal{O}(1)$ neighborhood of $\mathcal{M}^0$ which we can equip with a skeleton structure for the $\varepsilon = 0$ dynamics. This is done the vein of the fibration \((3.7)\) for our linear system. Here, nevertheless, translated copies of the stable/unstable manifolds $W^\pm_\varepsilon(z)$ will not suffice to foliate $\mathcal{F}_\varepsilon(z)$, since they (in general) do not form an invariant family. Our main tool is, instead, the Hartman–Grobman theorem, which establishes the existence of a diffeomorphism $\chi_\varepsilon : U \cap \mathcal{F}_\varepsilon(z) \to V$ between an $\ell$–dimensional neighborhood of $z$ lying on the fiber and a neighborhood $V \subset \mathbb{R}^\ell$ of the origin. This diffeomorphism maps the linear flow $\{\delta\phi^\varepsilon_t\}_{\tau \geq 0}$ to its nonlinear counterpart $\{\phi^\varepsilon_t\}_{\tau \geq 0}$. In particular, evolving a perturbation $\delta z \in \mathbb{R}^\ell$ under the linear flow and then mapping it to $\mathcal{F}_\varepsilon(z)$ using $\chi_\varepsilon$ produces the same result as first mapping it to $\mathcal{F}_\varepsilon(z)$ using $\chi_\varepsilon$ and then evolving it using the nonlinear flow on that fiber:

$$\forall \varepsilon > 0, \forall \tau \geq 0 \left[ \phi^\varepsilon_\tau(\chi_\varepsilon(\delta z)) = \chi_\varepsilon(\delta \phi^\varepsilon_\tau(\delta z)) \right], \text{ that is } \forall \tau \geq 0 \left[ \phi^0_\tau \circ \chi_\varepsilon = \chi_\varepsilon \circ \delta \phi^\varepsilon_\tau \right].$$

Under this diffeomorphism, the origin of the linear system is mapped to the fixed point $z$ of the nonlinear one, the stable/unstable eigenspaces are mapped to the stable/unstable manifolds $W^\pm_\varepsilon(z)$ and the invariant fibrations \((3.7)\) in $V$ are mapped to invariant fibrations in $U \cap \mathcal{F}_\varepsilon(z)$. It is this invariant fibration for the nonlinear system that organizes the dynamics in a neighborhood of $\mathcal{M}^0$.

3.6 A skeleton structure for the $\varepsilon > 0$ dynamics

In this section, we list the persistence results of GSPT for $\varepsilon$ in a neighborhood of zero and refer the reader to the references listed above for more information on the subject.

Persistence of $\mathcal{M}^0$. For $\varepsilon > 0$ in a neighborhood of zero, the manifold $\mathcal{M}^0$ perturbs to a manifold $\mathcal{M}^\varepsilon$ of the same dimension. This perturbed manifold is $\mathcal{O}(\varepsilon)$–close to $\mathcal{M}^0$, diffeomorphic to it, locally invariant, and the graph of a smooth function $h^\varepsilon : K \to \mathbb{R}^\ell$.

Fenichel theory, in fact, establishes the persistence of the stable/unstable manifolds $W^\varepsilon_+ (\mathcal{M}^0)$; the slow manifold $\mathcal{M}^\varepsilon$ is obtained as their intersection.

Persistence of $W^\varepsilon_\pm (\mathcal{M}^0)$. For $\varepsilon > 0$ in a neighborhood of zero, the manifolds $W^\varepsilon_0 (\mathcal{M}^0)$ perturb to manifolds $W^\varepsilon_\pm (\mathcal{M}^\varepsilon)$ of the same dimension. These perturbed manifold are $\mathcal{O}(\varepsilon)$–close to their $\varepsilon = 0$ counterparts, diffeomorphic to them, locally invariant, and the graphs of smooth functions $\eta^\varepsilon_\pm$ over subsets of the stable/unstable eigenspaces.

In fact, GSPT also characterizes the flow on these perturbed manifolds by means of establishing exponential contraction estimates (in forward/backward time, respectively, and towards $\mathcal{M}^\varepsilon$); see [Jones, pg. 71]. Additionally, it establishes perturbed fibrations of $W^\varepsilon_\pm (\mathcal{M}^\varepsilon)$ by means of $n$–dimensional families of $\ell_\pm$–fibers.

Persistence of $\{ W^\varepsilon_\pm(z) \}_{z \in \mathcal{M}^\varepsilon}$. Write $z^\varepsilon = (x, h^\varepsilon(x)) \in \mathcal{M}^\varepsilon$ and $z^0 = (x, h^0(x)) \in \mathcal{M}^0$. Then, for $\varepsilon > 0$ in a neighborhood of zero, the fiber $W^\varepsilon_\pm(z^\varepsilon)$ perturbs to a fiber $W^\varepsilon_\pm(z^0)$ of the same dimension. This perturbed fiber is $\mathcal{O}(\varepsilon)$–close to its $\varepsilon = 0$ counterparts and diffeomorphic to it. Each one of the two families $\{ W^\varepsilon_\pm(z^\varepsilon) \}_{z^\varepsilon \in \mathcal{M}^\varepsilon}$ is invariant as a family in the sense described earlier.
4.1 Relaxation oscillations in a predator–prey model

As a first application of GSPT, we will consider the appearance of relaxation oscillations in a rather generic predator–prey model.

We start by formulating the model. Letting \( x \) and \( y \) denote the populations of predators and prey, respectively, we write

\[
\begin{align*}
\dot{x} &= F(x, y; \varepsilon), \\
\varepsilon \dot{y} &= G(x, y; \varepsilon);
\end{align*}
\]

(4.1)

this is the slow formulation of our slow–fast, predator–prey system. A rescale of time via \( \tau = t/\varepsilon \) puts the system into the fast formulation,

\[
\begin{align*}
x' &= \varepsilon F(x, y; \varepsilon), \\
y' &= G(x, y; \varepsilon),
\end{align*}
\]

(4.2)

where \((\cdot)' = d \cdot /d\tau\). The region of interest is the (closed) quadrant \( \bar{R}_+ \times \bar{R}_+ = \{(x, y) | (x \geq 0) \land (y \geq 0)\} \), as negative populations are ecologically meaningless. In fact, we shall remain interested in a compact rectangle \([0, X] \times [0, Y]\) which, under suitable conditions, will need to be forward invariant: any initial condition \((x, y) \in [0, X] \times [0, Y]\) will need to yield a solution \( \{\phi_t(x, y)\}_{t \geq 0} \subset [0, X] \times [0, Y]\). Here, \( \varepsilon \in [0, \bar{\varepsilon}] \) is a small parameter \((\varepsilon \ll 1)\), dictating that the prey population evolves in a much faster timescale than the predator population.

Also, \( F \) and \( G \) are smooth functions, all derivatives of which remain bounded (i.e., are \( \mathcal{O}(1) \) as \( \varepsilon \downarrow 0 \)) over \([0, X] \times [0, Y] \times [0, \bar{\varepsilon}]\). Smoothness and time-invariance constitute our zeroth assumption on the nature of the model.

Our first assumption is that ‘there is no migration in the ecosystem,’ and hence a zero initial population of either prey or predator will remain zero at all times. Mathematically, this translates into (forward) invariance of the \( x \)- and \( y \)-axes, and in particular into the condition

\[
F(0, y; \varepsilon) = 0, \quad \text{for all } (y, \varepsilon) \in [0, Y] \times [0, \bar{\varepsilon}], \quad \text{and} \quad G(x, 0; \varepsilon) = 0, \quad \text{for all } (x, \varepsilon) \in [0, X] \times [0, \bar{\varepsilon}].
\]

It follows from this assumption that the origin is necessarily a fixed point (also called steady state or equilibrium), \( F(0, 0; \varepsilon) = G(0, 0; \varepsilon) = 0 \). Below, we use our ecological intuition and basic mathematical arguments to build up the dynamics of our system step-by-step. Then, following Hek, we show that systems of this sort can be expected to exhibit relaxation oscillations (also called slow–fast oscillations).

4.1.1 Detailed modeling

**Step 1: directionality of the \( x \)-component of the dynamics.** First, the populations of predators should diminish to zero, should prey become extinct (i.e., when \( y \equiv 0 \)). Therefore, \( F(x, 0; \varepsilon) < 0 \) for all \( x > 0 \) (cf. (4.1)). It now follows by continuity that

\[
F(x, y; \varepsilon) < 0, \quad \text{for all } 0 \leq y < y^*_x(x); \quad \text{in particular,} \quad F(x, y^*_x(x); \varepsilon) = 0.
\]

Here, \( y^*_x : [0, X] \rightarrow \mathbb{R} \) is some positive—possibly infinite—function. We now formulate explicitly our second assumption: \( y^*_x(x) = \sup\{y | F(x, y; \varepsilon) < 0\} \) is finite for all \((x, \varepsilon) \in [0, X] \times [0, \bar{\varepsilon}]\), and hence

\[
F(x, y; \varepsilon) < 0, \quad \text{for all } 0 \leq y < y^*_x(x); \quad F(x, y^*_x(x); \varepsilon) = 0; \quad \text{and} \quad F(x, y; \varepsilon) > 0, \quad \text{for all } y > y^*_x(x).
\]

This condition plainly formalizes the ecologically sensible condition that predator populations tend to decrease, for prey populations under a certain threshold, and tend to increase for prey populations over that threshold. Note, also, that it appears sensible to incorporate into our second assumption the requirement that \( y^*_x \) is non-decreasing. Indeed, the threshold prey population necessary for predators to thrive should not decrease with the predator population: the more the predators, the more prey they should require to be sustenance.

\[\footnote{1}{This happens if the prey adjusts to environmental conditions much faster than the predators, either because it responds much more quickly to changes in the available resources or because the predators can harvest prey at a very large rate.} \]

\[\footnote{2}{Built into this argument is the assumption that there are no alternative (called substitutable) food sources for the predators.} \]

\[\footnote{3}{Note that this behavior is not a necessity, if the response of predators to prey becomes saturated, since then the predator death rate may be uniformly larger than the reproduction rate, so that \( y^*_x = \infty \). In that case, \( F < 0 \) throughout \([0, X] \times [0, Y]\) and hence all predators are destined to die out. This case results into trivial dynamics, so we do not consider it here.} \]
We further assume that, for small predator populations and in the absence of prey, the decay rate of the population depends linearly on the population itself: $\delta x = \alpha \delta x$, for some $\alpha < 0$ and all small $\delta x$. Indeed, small populations lead to rare, negligible interactions between the individuals, and hence predator decay can be modeled as a Poisson process; coarse graining leads to the linear ODE above. Since $\partial_x F(0, 0; \varepsilon)$ and $F(0, 0; \varepsilon)$ yield that $\alpha = \partial_x F(0, 0; \varepsilon)$, we impose the condition $\partial_x F(0, 0; \varepsilon) < 0$. It follows by continuity that, for every $\varepsilon \in [0, \bar{\varepsilon}]$, there exists a relatively open set $D_\varepsilon \subset \mathbb{R}^2$ containing the origin such that

$$\partial_x F(x, y; \varepsilon) < 0, \text{ for } (x, y) \in D_\varepsilon.$$  

This implies, in particular, that $y^*_\varepsilon(0) > 0$. Since $y^*_\varepsilon$ is non-decreasing, then, $y^*_\varepsilon > 0$ uniformly over $[0, X]$.

These considerations fix the directionality of the $x-$component of the vector field throughout $[0, X] \times [0, Y]$, as well as its $x-$nullcline, namely, $\text{graph}(y^*_\varepsilon)$; see top-left panel of Fig. 7.

Step 2: the $y-$component of the dynamics. We now formulate our third assumption: ‘the life-sustaining resources available to prey are present but limited’. The first consequence of this is that $G(0, y; \varepsilon) > 0$, for $y$ in a right-neighborhood of zero: ecologically speaking, small prey populations should flourish in the absence of predators and presence of resources. It seems reasonable, further, to incorporate here the condition that the per capita growth rate of prey in the absence of predators, $G(0, y; \varepsilon)/y$, decreases with $y$: the larger the prey population, the fewer the resources available to each individual. In fact, we will assume that the derivative of $G(0, y; \varepsilon)/y$ is negative and bounded away from zero; factoring in the positivity of $G(0, y; \varepsilon)$ close to zero, one can show that there exists a unique value $\bar{y}_\varepsilon(0) > 0$ such that $G(0, \bar{y}_\varepsilon(0); \varepsilon) = 0$. Therefore, $(0, \bar{y}_\varepsilon(0))$ is an attracting fixed point for the predator-less model:

$$G(0, y; \varepsilon) > 0, \text{ for } 0 < y < \bar{y}_\varepsilon(0); \quad G(0, \bar{y}_\varepsilon(0); \varepsilon) = 0; \quad \text{and } G(0, y; \varepsilon) < 0, \text{ for } y > \bar{y}_\varepsilon(0).$$  

(See also top-right panel of Fig. 7.) By assumption, $G(0, :, \varepsilon)$ intersects zero transversally at $\bar{y}_\varepsilon(0)$ and hence $\partial_y G(0, \bar{y}_\varepsilon(0); \varepsilon) < 0$. Consequently, prey populations approach the carrying capacity $\bar{y}_\varepsilon(0)$, corresponding to the prey population that the available, limited resources can sustain in the absence of predators, exponentially in time. It also now automatically follows that the origin is unstable.

We finally assume that, in the absence of predators, small prey populations follow Malthusian dynamics: $\delta y = \beta \delta y$, for some $\beta > 0$ and small $\delta y$. Here again, $\partial_y G(0, 0; \varepsilon) = 0$ yield that $\beta = \varepsilon^{-1} \partial_y G(0, 0; \varepsilon)$, and hence $\beta = \partial_y G(0, 0; \varepsilon) > 0$. It follows by continuity that, for every $\varepsilon \in [0, \bar{\varepsilon}]$ there exists a relatively open set $E_\varepsilon \subset \mathbb{R}^2$ containing the origin such that

$$\partial_y G(x, y; \varepsilon) > 0, \text{ for } (x, y) \in E_\varepsilon.$$  

A corollary of paramount importance to this statement is that $\partial_y G(x, 0; \varepsilon) > 0$, for $x$ in some interval $[0, x^*_\varepsilon)$; that is, the (invariant) $x-$axis is normally repelling in a neighborhood (possibly unbounded) of the origin. Here, we will assume that $x^*_\varepsilon < X$, and in particular that $x^*_\varepsilon$ is finite. These results determine the direction of the $y-$component of the vector field on the $y-$axis, as well as in an $\mathcal{O}(1)$ neighborhood of $[0, x^*_\varepsilon] \times \{0\}$. By assuming that $\partial_y G(x, 0; \varepsilon) < 0$, for $x > x^*_\varepsilon$, we fix the flow in a neighborhood of the entire interval $[0, X]$ on the $x-$axis. Note that this last assumption on $\partial_y G(x, 0; \varepsilon)$ is quite natural, as is the stronger one that $\partial_y G(x, 0; \varepsilon)$ is decreasing. Indeed, the larger the predator population small prey populations are exposed to, the less rapidly they should grow—if at all.

Step 3: upper branch of the slow manifold $\mathcal{M}_0$. We assumed above that $\partial_y G(0, \bar{y}_\varepsilon(0); \varepsilon) < 0$, whence also $\partial_y G(0, \bar{y}_0(0); 0) < 0$, and remarked that this assumption fits in well with the monotonicity of $G(0, y; \varepsilon)/y$. It follows by the Implicit Function Theorem that the equation $G(x, y; 0) = 0$ can be solved to yield $y = \bar{y}_0(x)$, for all $x \in [0, x_f]$ and some $x_f > 0$ (possibly infinite). Additionally (and by continuity),

$$\partial_y G(x, \bar{y}_0(x); 0) < 0, \text{ for all } x \in [0, x_f], \quad \text{while } \partial_y G(x_f, y_f; 0) = 0 \quad (\text{where we write } y_f := \bar{y}_0(x_f)).$$

4The reader should be able to show that this agrees with—but does not follow from—the condition imposed above: that $F(x; 0; \varepsilon) < 0$, for $x > 0$.

5The minimal such model models $G(0, y; \varepsilon)/y$ linearly: $G(0, y; \varepsilon) = y(c - y)$, for some constant $c > 0$—see Heck Example 2.1. This is called the logistic model.

6Here also, the reader can show that this condition agrees (but cannot be derived from) the condition $G(0, y; \varepsilon) > 0$, for $y$ in a neighborhood of zero.

7This amounts to assuming that a predator population kept at a constant, high enough value can lead prey to total extinction.
This result implies that the fast formulation (4.2) possesses, for \( \varepsilon = 0 \), an entire branch \( \mathcal{M}_0 \) of normally attracting fixed points: namely, the graph of the function \( \bar{y}_0 \), see bottom-left panel of Fig. 7. The value \( \bar{y}_0(x) \) can be thought of as the carrying capacity for a constant predator population \( x \). Since predators harvest on prey, it is sensible to assume that this carrying capacity decreases with \( x \), i.e., that \( \bar{y}_0 \) is a decreasing function. Our fourth assumption is that \( 0 < x_f < X \); that \( 0 < \bar{y}_0(x_f) < y_f \); that \( G(x; y; 0) = 0 \) has no solutions for \( x > x_f \); and that, along every horizontal line \( y = \text{const.} = c \in [y_f, \bar{y}_0(0)] \), the solution \( x = \bar{y}_0^{-1}(c) \) of \( G(x, c, 0) = 0 \) is unique. We include in this assumption a final demand—namely, that \( \partial_x G(x_f, y_f; 0) \neq 0 \), as a consequence of which the branch of fixed points we identified above must fold at \( x_f \), i.e., \( \bar{y}_0'(x_f) = -\infty \) [why?].

The first part of our assumption (involving \( x \)), as well as the second and the last (the branch folds, and does so before hitting either zero or the \( x \)--nullcline) are rather arbitrary and serve the purpose of ‘spicing up’ the dynamics; hence, they should be interpreted as focusing into a particular parameter regime. In particular, an intersection of the \( x \)--nullcline with this upper branch of \( \mathcal{M}_0 \) would lead to a (globally) stable equilibrium, which we want to avoid. Further, the assumption that \( \mathcal{M}_0 \) folds will yield the relaxation oscillation we are after. The third part, on the other hand, is rather sensible; in fact, one expects that \( G(\cdot, y; 0) \) is a decreasing function for every fixed value of \( y \), since the rate at which any given prey population \( y \) grows should decrease with the predator population \( x \) to which it is exposed. Should this be true, the third-through-fifth parts of our assumption follow immediately by virtue of \( G(x, y; 0) > 0 \) (for \( 0 < y < \bar{y}_0(0) \)) and as long as \( \partial_x G(\cdot, y; 0) < 0 \) remains bounded away from zero over \([0, X]\) (and for each \( y \in [0, Y] \)).

As a result of the above considerations, we conclude that

\[ G(x, y; 0) > 0 \in \{ (x, y) | [y_f < y < \bar{y}_0(0)] \cap [0 < x < \bar{y}_0^{-1}(y)] \} \]  
\[ G(x, y; 0) < 0 \in \{ (x, y) | [y_f < y < \bar{y}_0(0)] \cap [\bar{y}_0^{-1}(y) < x < X] \} . \]  

The separatrix between these two regions is the upper branch \( \mathcal{M}_0 \) of the slow manifold. This characterization of the sign of \( G \) fixes the directionality of the \( y \)--component of the vector field in the rectangle \([0, X] \times [y_f, \bar{y}_0(0)]\). Also, the characterization of the sign of \( F \) we offered above fixes the flow (namely, towards increasing \( x \)) on this branch of the slow manifold.

**Step 4: lower branch of the slow manifold \( \mathcal{M}_0 \).** Next, we show that there exists a second, lower branch \( \mathcal{M}_0 \) of \( \mathcal{M}_0 \) which connects \((x_f, y_f)\) to \((x_0, 0)\). First, (4.4) yields that \( G(0, \cdot; 0) > 0 \) over \((0, \bar{y}_0(0))\). At the same time, (4.7) yields that \( G(x, \cdot; 0) < 0 \) over \((y_f, \bar{y}_0(0)) \subset (0, \bar{y}_0(0))\), for any \( x_f < x < X \). Since, by assumption, \( G(x, \cdot; 0) \) has no zeros for \( x > x_f \), it follows that \( G(x, \cdot; 0) > 0 \) over the entire domain \((x_f, X) \times (0, Y)\). By continuity, along each line \( y = \text{const.} = c \in (0, y_f) \), there exists a value \( x_0(c) \) such that \( G(x_0(c), c, 0) = 0 \). Each such point \((x_0(c), c)\) is a point on the lower branch \( \mathcal{M}_0 \) of the slow manifold \( \mathcal{M}_0 \), see bottom-right panel of Fig. 7.

As part of our fifth assumption, we impose that the value \( x_0(c) \) is unique—recall also our reasoning for the analogous assumption imposed on the upper branch \( \mathcal{M}_0 \). Further, we assume that \( x_0 \) is an increasing function and \( \partial_y G < 0 \) on it. Under this last assumption, \( x_0 \) is invertible and \( y_0 := x_0^{-1} \) is an increasing function defined in a left-neighborhood of \( x_f \), the graph of which is \( \mathcal{M}_0 \). To shed more light into the significance of this fifth assumption, we remark that \( G(x, y_0, 0) = 0 \)—the vertical component of the vector field vanishes on \( \mathcal{M}_0 \)—and \( G \) changes sign across it (by virtue of \( \partial_y G \neq 0 \), which means that \( G \) crosses zero transversally across \( \mathcal{M}_0 \)). It follows that, for any fixed predator population \( x_c \) smaller than and sufficiently close to \( x_f \), initial prey populations larger than \( y_c(x_c) \) grow towards the stable steady state \( \bar{y}_0(x_c) \), while initial prey populations smaller than \( y_c(x_c) \) decay to zero (in particular, \( \mathcal{M}_0 \) is normally repelling). Hence, the value \( y_c(x_c) \) forms a threshold: higher initial populations limit up to the nontrivial steady state: lower initial populations vanish. The assumption that \( x_0 \)—and hence also \( y_0 \)—is increasing means that this threshold grows with the predator population—which is ecologically sound, since it should take higher prey populations to overcome higher predator populations. In particular, our monotonicity assumption excludes a bistable scenario, in which \( \mathcal{M}_0 \) would fold again and yield another, lower branch of stable steady states. Since, in general, bistable scenarios cannot be excluded, this choice should again be viewed as focusing into a specific parameter regime.

It remains to determine the left-neighborhood of \( x_f \) over which \( y_c(x) \) is defined, that is, its left endpoint \( x_0(0) \). As we showed in Step 2, the \( x \)--axis is normally repelling in a neighborhood \((0, x_0)\); since \( G(x, y; 0) < 0 \) for \( 0 < y < y_0(x) \), the \( x \)--axis is normally attracting for all values of \( x \) in the domain of definition of \( y_0(x) \). As a result, this domain of definition cannot extend below (i.e., to the left of) \( x_0 \), that is, \( x_0(0) \geq x_0 \). Further, assuming that \( \mathcal{M}_0 \) intersects the \( x \)--axis transversally at \((x_0(0), 0)\)—equivalently, that \( y_0'(x_0(0)) > 0 \)—we find
that both $\partial_x G$ and $\partial_y G$ are zero at that point [why?]. Since $x^*_0$ is the unique zero of $\partial_y G(x, 0; 0)$ by assumption, it follows that $y^*_0$ is defined exactly over $[x_0, x_f]$; consequently, $y^*_0(x_0) = 0$ and the lower branch $\mathcal{M}_0$ ends up at the point $(x^*_0, 0)$. Note that, necessarily then, $x^*_0 < x_f$.

**Step 5: a nontrivial unstable equilibrium.** Finally, we remark that the system necessarily possesses a nontrivial equilibrium $(x_0, y_0) \in \mathcal{M}_0$, as $y_0(x_0) = 0 < y^*_0(x_0)$ and $y_0^*(x_f) = y_f > y^*_0(x_0)$. This equilibrium is also necessarily unstable, since $\mathcal{M}_0$ is normally repelling. Additionally, the flow on $\mathcal{M}_0$ is towards decreasing $x$ to the left of the equilibrium and towards increasing $x$ to its right—this fixes the flow on the rest of the slow manifold.

**Remark.** Note that (4.3) and (4.5) also imply [why?] that (4.1) can be rewritten as

$$
\begin{align*}
\dot{x} &= xf(x, y), \\
\dot{y} &= yg(x, y).
\end{align*}
$$

(4.8)

Here, the functions $f$ and $g$ satisfy the same smoothness assumptions as $F$ and $G$. Additionally,

$$
\begin{align*}
f(0, y; \varepsilon) < 0, & \quad \text{for } y \in [0, y^*_x), \\
g(x, 0; \varepsilon) > 0, & \quad \text{for } x \in [0, x^*_x).
\end{align*}
$$

By continuity, and for any $0 \leq \varepsilon \leq \varepsilon$, there exist sets $D_{\varepsilon}$ and $E_{\varepsilon}$ such that

$$
\begin{align*}
f(0, y; \varepsilon) < 0, & \quad \text{for } (x, y) \in D_{\varepsilon}, \\
g(x, 0; \varepsilon) > 0, & \quad \text{for } (x, y) \in E_{\varepsilon}.
\end{align*}
$$

The signs of $f$ and $g$ can be deduced by our analysis above [do it!].

### 4.1.2 The relaxation limit cycle

Having constructed our rather generic predator-prey, fast–slow model, we now turn to identifying a limit cycle in it. This limit cycle is schematically constructed (to leading order) in Fig. [5], as can be seen from that sketch, it consists of alternating fast and slow components. Fenichel theory guarantees that all three branches of the slow manifold—$\mathcal{M}_0$, $\mathcal{M}_0$, and any compact subset of the positive $x$–axis perturb for sufficiently small values of $\varepsilon > 0$, since they are normally hyperbolic. Note that this includes neither the fold point $(\bar{x}, \bar{y}(\bar{x}))$ nor the point of intersection $(x^*_0, 0)$, as $\mathcal{M}_0$ loses normal hyperbolicity at both of these points. In particular, one has to exclude $O(1)$—neighborhoods around each one of these points to assure persistence of the slow manifold branches.

The perturbed branches have the same stability properties as their $\varepsilon = 0$ counterparts and the evolution on them is governed, to leading order, by the dynamics of the $\varepsilon = 0$ slow system. Hence, for any sufficiently small $\varepsilon > 0$, any fixed initial condition $\varepsilon$–close to $\mathcal{M}_0$ generates a trajectory which stays $\varepsilon$–close to it (by invariance and normal attractivity of $\mathcal{M}_\varepsilon$, which is $\varepsilon$–close to $\mathcal{M}_0$). Since $\mathcal{M}_\varepsilon$ is $\varepsilon$–close to $\mathcal{M}_0$, the leading order component of the trajectory is $y_0(\cdot) = \bar{y}_0(x_0(\cdot))$; here, the evolution of $x_0$—the leading order component of $x$—is determined by the system

$$
\begin{align*}
\dot{x}_0(t) &= F(x_0(t), \bar{y}_0(x_0(t)); 0) \quad \text{and} \quad y_0(t) = \bar{y}_0(x_0(t)),
\end{align*}
$$

up to the point $\bar{t}$ where $x(\bar{t}) = x_f$. (A leading order formula for $t_f$ may be derived from the equation $x_0(\bar{t}) = x_f$.) Following that time instant, the trajectory is captured by the fast dynamics and drawn—to leading order vertically—to the $x$–axis according to the system

$$
\begin{align*}
x_0(\tau) &= x_f \quad \text{and} \quad \dot{y}_0(\tau) = G(x_f, y_0(\tau); 0), \quad \text{where} \quad \tau = t/\varepsilon.
\end{align*}
$$

Following this fast descent to the point $(x_f, 0)$ (always to leading order), the trajectory is trapped in an $O(\varepsilon)$ neighborhood of the $x$–axis—note that this axis remains invariant for $\varepsilon > 0$ and thus represents the lowest branch of $\mathcal{M}_\varepsilon$ also for $\varepsilon > 0$. The trajectory proceeds towards decreasing values of $x$ following, to leading order, the dynamics

$$
\begin{align*}
\dot{x}_0(t) &= F(x_0(t), 0; 0) \quad \text{and} \quad y_0(t) = 0.
\end{align*}
$$

(4.9)

We remark here that the next-order term of the $y$–component is attracted towards zero according to the linearized normal dynamics around the $x$–axis, and thus the trajectory is attracted to that axis up to within
an \( \varepsilon \)-neighborhood of \( x_0^* \) and repelled afterwards. At a certain point \( x_l < x_0^* \), the trajectory ‘lifts off’ \( \mathcal{M}_\varepsilon \) and follows the fast dynamics,

\[
x_0(\tau) = x_l \quad \text{and} \quad y'_0(\tau) = G(x_l, y_0(\tau); 0),
\]

towards the point \( x_l, \bar{y}_0(x_l) \). It then follows the slow dynamics towards \( x_f \) as described in the beginning of this paragraph.

Naturally, a more careful analysis needs to work locally in neighborhoods around the fold and intersection points, where \( \mathcal{M}_\varepsilon \) is not defined, and establish that the trajectory indeed enters and leaves those neighborhoods. Additionally, the leading order solutions obtained for the four different parts must be matched properly. Note, also, that the initial condition for the limit cycle must lie below \( \mathcal{M}_\varepsilon \) [why?] and that a proper proof of its existence can be given with the help of a Poincaré section transversal to \( \mathcal{M}_\varepsilon \). Nevertheless, our—very tractable—analysis here provides a feeling for the location of (and the dynamics on) this limit cycle, up even to the location of the lift-off point \( x_l \).

To calculate the location of that lift-off point, let us define ‘touch-down’ and ‘lift-off’ as the moments \( t_f \) and \( t_l \) that the trajectory enters and leaves the rectangle \([x_f, x_l] \times [0, y_l]\), respectively. Here, \( y_l = \varepsilon \bar{y}_1 \) and \( \bar{y}_1 \) is a fixed value. Now, the first leg of (4.9) implies that \( x \) is a monotone function of \( t \) (recall that \( F < 0 \) on the \( x \)-axis). Hence, it can be used to reparameterize the solution \((x(\cdot), y(\cdot))\) in the time interval \([t_f, t_l]\) according to the ODE

\[
\frac{dy_0(x_0)}{dx_0} = \frac{y_0(x_0)G_y(x_0, 0; 0)}{\varepsilon F(x_0, 0; 0)},
\]

where we have Taylor-expanded the right member with respect to \( y \), as \( y = \mathcal{O}(\varepsilon) \) throughout the rectangle \([x_f, x_l] \times [0, y_l]\), and recalled that \( G(x_0, 0; 0) = 0 \) for every \( x \geq 0 \). (Note the slight abuse in notation involved in writing \( y_0(x_0) \) instead of \( y_0(t(x_0)) \).) Integrating from \( x_0 = x_f \) to \( x_l \), we find then an algebraic equation for \( x_l \) with a unique [why?] solution,

\[
0 = \varepsilon \left[ -\ln|y_0| \right]_{y_l}^{y_l} = \int_{x_f}^{x_l} \frac{G_y(x; 0; 0)}{F(x, 0; 0)} \, dx.
\]

**Discussion.** In the above 2–D example, we used phase plane analysis together with GSPT to (formally) construct a relaxation limit cycle. A large part of the construction can be made rigorous using persistence results from GSPT—notably, those for the persistence of a slow manifold, of its stable and unstable manifolds, and of a fast fibration of each one of these manifolds. The flow around the (fold and intersection) points where GSPT breaks down, on the other hand, requires further local analysis.

We finally remark that the relaxation oscillation that we constructed here exhibits catastrophic properties in that the prey population varies abruptly when the predator population reaches the critical values \( x_f \) and \( x_l \) (to leading order). Naturally, this is far from surprising given the configuration of the phase space, which exhibits a fold—the hallmark of many catastrophic events. The reader should note, in particular, that the \( \varepsilon = 0 \) phase space for the fast system is, in fact, the Cartesian product of a 1–D phase space with \( \mathbb{R}_+ \) (the space in which \( x \) takes values). Seen in that light, \( x \) acts as a parameter and the \( (x \text{–dependent}) \) phase line undergoes a saddle–node (fold) bifurcation at \( x_f \), as long as \( f \) is quadratic (which it generically is). (This also casts a different light on the condition \( \partial_x G(x_f, y_f; 0) \neq 0 \), as this becomes one of the two standard conditions associated with this type of bifurcation.)

As mathematically unsurprising as this behavior is, it has severe ecological significance because it showcases an all too sudden disappearance of a prey species. Indeed, one would have a hard time predicting this phenomenon using a timeseries \( \{y(t_n)\}_{1 \leq n \leq N} \) for the prey population, as the observed behavior is most certainly not analytic and hence cannot be captured by ordinary data fitting. Additionally, the lift-off at \( x_l \) exhibits all the properties pertaining to a delayed bifurcation. Note, in particular, that rebuilding a substantial prey population requires a predator population much smaller than (the fold point) \( x_f \) where the catastrophe occurs: for any value \( x \in (x^*_f, x_f) \) of the population, the stable prey population corresponding to the upper branch \( M_0 \) of the slow manifold remains unattainable, as it is separated from the \( x \)-axis by the unstable branch. As we explained above (cf. (4.10)), \( x_l \) is separated from \( x_0^* \)—where the \( x \)-axis reverses its normal stability type—by

---

8 The following is an indirect indication of the fact that, for \( \varepsilon > 0 \), no counterpart of \( M_0 \) exists around the fold point to guide the dynamics: first, recall that the part of the trajectory lying \( \varepsilon \)-close to \( M_0 \) lies below it. Now, should \( M_0 \) perturb close to the fold point, in the sense of an invariant manifold joining the perturbed counterparts of \( M_0 \) and \( \bar{M}_0 \), the limit cycle would need to cross it in order to approach the \( x \)-axis. This cannot occur, as it would violate uniqueness of solutions for our system, which offers us the desired contradiction.
an $O(1)$ distance which is necessary to allow the trajectory to ‘peel off’ after it spent the interval $(x_0^*, x_f)$ getting ‘compressed’ towards the $x$-axis. Most interesting of all, the population remains close to the unstable $x$-axis past $x^*_\varepsilon$ where the stability changes and although the only stable prey populations are those found on the upper branch of $M_0$. In short, this means that the predator population must be reduced way below the level $x_f$ at which the catastrophe occurred for the prey population to be built up again.

### 4.2 Relaxation oscillations in Rayleigh’s equation

As a second example of relaxation oscillations, we consider the well-known and studied Rayleigh’s equation,

\[
\varepsilon \ddot{y} + \left( \frac{1}{3} \dot{y}^2 - 1 \right) \dot{y} + y = 0,
\]

which can be thought of as modeling a linear spring with nonlinear friction/excitation (compare to our linear model in Sect. 2.2) and is the prototypical example of a Liénard equation. In particular, the nonlinearity corresponds to damping, for $|\dot{y}| < \sqrt{3}$, and to excitation, for $|\dot{y}| > \sqrt{3}$. As we will see in this section, this equation possesses a solution corresponding to a relaxation limit cycle. In next week’s lecture, we will see that this equation possesses a periodic orbit not only for arbitrarily small but also for arbitrarily large $\varepsilon$. In fact, it can be proven that Rayleigh’s equation possesses a limit cycle for every positive value of $\varepsilon$. [In our setting, it is instructive to try to show that (4.11) has a limit cycle by constructing a bounding box in the phase plane and applying the Poincaré–Bendixson theorem.]

To construct the relaxation oscillation as in the last section, we write the singularly perturbed ODE (4.11) as a system of two first-order ODEs,

\[
\begin{align*}
\dot{y} &= z, \\
\varepsilon \dot{z} &= -y + z - \frac{1}{3} z^3.
\end{align*}
\]

For $\varepsilon = 0$, this system possesses a manifold $M_0$ of fixed points together with a vertical fibration over it, see Fig. 9. The corresponding slow system yields the flow on $M_0$ and reveals that the origin is a source. As
we discussed in last week’s lecture, the skeleton structure of the phase space for \( \varepsilon > 0 \) may be obtained by concatenating the \( \varepsilon = 0 \) fast and slow dynamics. This idea leads, here again, to the leading order construction of the limit cycle sketched in that same figure. Here also, a local analysis must be carried out around either fold point where the slow manifold is not expected to perturb to establish that trajectories near \( \mathcal{M}_0 \) enter a neighborhood of either of these points, get captured by the fast dynamics, and exit it. Our sketch also suggests the amplitude (to leading order) of the oscillation, as well as that the limit cycle is (globally) stable. The period of the oscillation can be also obtained to leading order in the following way: first, the fast transients last an \( o(1) \) amount of time and thus can be omitted to leading order—hence, the period of the limit cycle equals, to leading order, the sum of the amounts of time the cycle spends on either stable branch of \( \mathcal{M}_0 \). By the antisymmetry of the vector field, these two amounts are equal and thus the leading order term for the period equals the time it takes the trajectory to move from \((-2/3, 2)\) to \((2/3, 1)\). Now, since \( \mathcal{M}_0 = \text{graph}(z - z^3/3) \) and \( \dot{y} = z \), we find

\[
z = \dot{y} = (1 - z^2) \dot{z},
\]

whence we obtain the period to leading order,

\[
T = 2 \int_0^{T/2} dt = 2 \int_2^1 \frac{1 - z^2}{z} \, dz + o(1) = 3 - 2 \ln 2 + o(1).
\]

---

\( ^9 \)In fact, an analysis around either one of these two fixed points is enough, as the vector field is antisymmetric with respect to reflections about the origin.
Figure 9: Concatenation of the \( \varepsilon = 0 \) fast and slow dynamics in the phase space for Rayleigh’s equation (4.11). The slow (blue) and fast (red) components of the leading order relaxation oscillation are also sketched.

Lecture no.5

March 05, 2013

5.1 The method of strained coordinates

[See also [Kevorkian & Cole, Sect. 4.1-2] for a detailed discussion of the method of strained coordinates, a.k.a., Poincaré–Lindstedt method.] At this point, it might seem that we have all we need to deal with perturbed ODEs: a procedure to approximate inner solutions (if any exist) and another one for outer solutions (both to arbitrary asymptotic accuracy), and even a way to combine (match) them into a uniformly valid asymptotic expansion. Additionally, in the last two lectures, we have examined a geometric framework which, among other things, allows us to go as far as predicting where layers will turn up and how many (if any) solutions satisfying our initial/boundary value problem may be expected to exist.

Nevertheless, a large part of the machinery we developed in the lectures up to this one can only deal with problems in which the independent variable takes values in a bounded set. Consider, for example, the regularly perturbed problem we treated in Section 2.1: both the two-term asymptotic expansion we obtained there (cf. (2.6) and (2.6)) and the exact solution (2.2) grow unboundedly (in fact, exponentially) as \( x \to \infty \), albeit the difference between them also grows exponentially, so that it remains \( O(\varepsilon^2) \) only over a finite time horizon. Even worse, consider the problem in exercise 01(b) of Homework #02, which deals with a regularly perturbed harmonic oscillator,

\[
\ddot{y}(t) + (1 + \varepsilon) y(t) = 0, \quad \text{with} \quad y(0) = 0 \quad \text{and} \quad \dot{y}(0) = 1.
\]  (5.1)

As we saw, the two-term asymptotic expansion we derived only remains uniformly valid over a finite time, although both it and the actual solution remain bounded. Indeed, postulating the asymptotic expansion

\[
y(t) = \sum_{n \geq 0} \varepsilon^n y_n(t),
\]

substituting into (5.1), and collecting terms of equal order in \( \varepsilon \), we obtain, up to and including terms of \( O(\varepsilon) \),

\[
\begin{align*}
\ddot{y}_0 + y_0 &= 0, \quad \text{with} \quad y_0(0) = 0 \quad \text{and} \quad \dot{y}_0(0) = 1, \\
\ddot{y}_1 + y_1 &= -y_0, \quad \text{with} \quad y_1(0) = 0 \quad \text{and} \quad \dot{y}_1(0) = 0.
\end{align*}
\]  (5.2)

The first of these problems yields \( y_0(t) = \sin t \). Substituting into the second ODE and using variation of constants [Bender & Orszag, Section 1.5], we obtain for \( y_1 \) the general solution

\[
y_1(t) = \left[ C_+ + \int_0^t \sin^2(s) \, ds \right] \cos(t) + \left[ C_- - \int_0^t \cos(s) \sin(s) \, ds \right] \sin(t).
\]
Here, $C_{\pm}$ are arbitrary constants. Evaluating the integrals and imposing the initial conditions, we finally find

$$y_1(t) = \frac{1}{2} t \cos t - \frac{1}{2} \sin t.$$  

It now becomes evident that the term $\varepsilon y_1$ becomes commensurate with the leading order term $y_0$ in the regime $t = O(1/\varepsilon)$; in other words, the asymptotic expansion stops being well-ordered in that regime, see also Fig. [10] (It can be easily shown that all subsequent terms in the asymptotic expansion become commensurate with $y_0$; the problem is not limited to $\varepsilon y_1$.)

In this particular case, the result can be anticipated. Indeed, the forcing term in [5.3] is oscillatory with frequency equal to the system’s eigenfrequency; hence, one should expect resonance. (The way this manifests itself mathematically is through the term $\int_0^t \sin^2(s) \, ds$ which becomes unbounded as $t \to \infty$—note that the integrand is periodic and non-negative!) From yet another point of view, our linear model can be solved to yield the exact solution

$$y(t) = \frac{1}{\sqrt{1 + \varepsilon}} \sin \left( \sqrt{1 + \varepsilon} \, t \right).$$  

Expanding the $\varepsilon$–dependent terms in powers of $\varepsilon$, we obtain

$$y(t) = \sin(t) + \varepsilon \left[ \frac{1}{2} t \cos t - \frac{1}{2} \sin t \right] + O(\varepsilon^2);$$

this matches the asymptotic expansions we derived above (the same applies to all orders in $\varepsilon$), and which is only well-ordered for finite times. It is interesting to note here that, although the asymptotic expansion is not well-ordered past $t = O(1/\varepsilon)$, the expansion derived above remains, in fact, convergent for all $t \in \mathbb{R}$. This is far from surprising: indeed, the only $t$–dependent term in [5.4] is

$$\sqrt{1 + \varepsilon} \, t = \left( 1 + \frac{1}{2} \varepsilon + O(\varepsilon^2) \right) t;$$

the (asymptotically) largest term in this asymptotic expansion is $t$, while $\varepsilon t$ is the (asymptotically) largest perturbative term. For $t \ll 1/\varepsilon$, this last term stops being perturbative and becomes commensurate—or even asymptotically larger—than $t$, the value around which we expand. Hence, it stands to reason that the Taylor expansion we produce is not an asymptotic expansion of the actual solution; this, nevertheless, is unrelated to whether it is convergent or not.

Since the asymptotic series we produced is convergent, its terms must progressively get smaller. Nevertheless, the bigger the time interval we wish to approximate the solution over, the more terms we have to include in the asymptotic expansion. If we allow the time interval to become unbounded, every truncated series will diverge from the actual solution, as the series is not uniformly convergent over any unbounded time interval. From a phenomenological point of view, the exact solution has a period equal to $2\pi/\sqrt{1 + \varepsilon}$, whereas the asymptotic expansion involves $2\pi$–periodic functions; it is only to be expected that the phase difference accumulated at each period grows with time. This consideration also suggests the remedy, which lies at the heart of the method of strained coordinates: first, we change the timescale (rescale time) via $\tau = \omega(\varepsilon) \, t$, where $\omega(\varepsilon) = 1 + \sum_{n \geq 1} \varepsilon^n \omega_n$ is the new timescale. (The leading order term in this expansion is chosen equal to one for definiteness.) [Make sure you understand why we have the freedom to do this without compromising the applicability of the method and what are the significance and consequences of this choice.] Then, we express the solution in terms of the new temporal variable $\tau$ and write $y(t) = \psi(\tau)$. The initial value problem that $\psi$ has to satisfy is

$$\omega^2(\varepsilon) \psi''(\tau) + (1 + \varepsilon) \psi(\tau) = 0, \quad \text{with} \quad \psi(0) = 0 \quad \text{and} \quad \psi'(0) = \omega^{-1}(\varepsilon).$$

Expanding the new dependent variable as $\psi(\tau) = \sum_{n \geq 0} \varepsilon^n \psi_n(\tau)$ and following the same procedure as before $^1$ we obtain

$$\begin{align*}
\psi''' + \psi_0 &= 0, \quad \text{with} \quad \psi_0(0) = 0 \quad \text{and} \quad \psi_0'(0) = 1, \\
\psi'' + \psi_1 &= -2\omega_1 \psi_0' - \psi_0, \quad \text{with} \quad \psi_1(0) = 0 \quad \text{and} \quad \psi_1'(0) = -\omega_1.
\end{align*}$$

$^1$Naturally, this problem can be solved explicitly, just like our original problem could; we derive the asymptotic expansion of the solution rather than the solution itself solely to demonstrate the method.
Figure 10: The exact solution—$y$, in blue—and the two-term expansion—$y_0 + \varepsilon y_1$, in red—corresponding to regularly perturbed harmonic oscillator (5.1). The amplitude lines $y = \pm 1/\sqrt{1 + \varepsilon}$ corresponding to the maxima and minima of the exact solution are also shown.

The first of these problems is identical to that for $y_0$, and hence $\psi_0(\tau) = \sin(\tau)$. Substituting into the problem for $\psi_1$, we find

$$\psi_1''(\tau) + \psi_1(\tau) = (2\omega_1 - 1)\sin(\tau), \quad \text{with } \psi_1(0) = 0 \quad \text{and} \quad \psi_1'(0) = -\omega_1. \quad (5.5)$$

Here also, the forcing is resonant and hence it will lead to an unbounded solution if not eliminated. To avoid this, we do eliminate it by making the choice $\omega_1 = 1/2$. Substituting into (5.5) and solving, subsequently, the problem, we find $\psi_1(\tau) = -\sin(\tau)/2$, so that

$$\psi(\tau) = \left[1 - \frac{\varepsilon}{2} + \mathcal{O}(\varepsilon^2)\right]\sin(\tau), \quad \text{or equivalently} \quad y(t) = \left[1 - \frac{\varepsilon}{2} + \mathcal{O}(\varepsilon^3)\right]\sin \left[(1 + \varepsilon/2 + \mathcal{O}(\varepsilon^2))t\right].$$

Note that the asymptotic expansion we obtain in this way corresponds to asymptotically expanding, in powers of $\varepsilon$, the solution’s amplitude and eigenfrequency but not the sinusoidal term itself.

### 5.2 The multiscale method

[See Holmes, Sect. 3.1-2 for a detailed discussion of the multiscale method.] Looking back at the idea underpinning the method of strained coordinates, it becomes intuitively clear that the method should do well whenever a sole timescale is involved. This raises the question of how can one treat problems involving multiple timescales. To elucidate the problem, we deal with the example presented in Holmes, Sect. 3.1-2,

$$\ddot{y}(t) + \varepsilon \dot{y}(t) + y(t) = 0, \quad \text{with} \quad y(0) = 0 \quad \text{and} \quad \dot{y}(0) = 1, \quad (5.6)$$

which models a weakly damped harmonic oscillator. A standard application of the Liouville transform, $Y(t) = e^{\frac{\varepsilon t}{2}}y(t)$, puts the problem in the form

$$\ddot{Y}(t) + \left(1 - \frac{\varepsilon^2}{4}\right)Y(t) = 0, \quad \text{with} \quad Y(0) = 0 \quad \text{and} \quad \dot{Y}(0) = 1.$$

This is the model problem (5.1) in disguise: the asymptotically small parameter $\varepsilon$ appearing in that model has been replaced by $-\varepsilon^2/4$ in this one. Consequently, the method of strained coordinates can be employed to deal with (5.6), at least after the Liouville transform has been employed. Can it deal equally well with the problem before the application of that transform? To find out, we introduce the new temporal variable $\tau = \omega(\varepsilon)t$ and dependent variable $y(t) = \psi(\tau)$ to recast the problem in the form

$$\omega^2(\varepsilon)\psi''(\tau) + \varepsilon \omega(\varepsilon) \psi'(\tau) + \psi(\tau) = 0, \quad \text{with} \quad \psi(0) = 0 \quad \text{and} \quad \psi'(0) = \omega^{-1}(\varepsilon).$$
Postulating the asymptotic expansions $\omega(\varepsilon) = 1 + \sum_{n \geq 1} \varepsilon^n \omega_n$ and $\psi(\tau) = \sum_{n \geq 0} \varepsilon^n \psi_n(\tau)$ and working as before, we obtain

$$
\psi''_0 + \psi_0 = 0, \quad \text{with } \psi_0(0) = 0 \quad \text{and} \quad \psi'_0(0) = 1,
$$

$$
\psi''_1 + \psi_1 = -2\omega_1 \psi''_0 - \psi'_0, \quad \text{with } \psi_1(0) = 0 \quad \text{and} \quad \psi'_1(0) = -\omega_1.
$$

The zeroth-order initial value problem has been solved before, $\psi(\tau) = \sin(\tau)$. Substituting into the first-order problem yields

$$
\psi''_1 + \psi_1 = 2\omega_1 \sin(\tau) - \cos(\tau), \quad \text{with } \psi_1(0) = 0 \quad \text{and} \quad \psi'_1(0) = -\omega_1.
$$

A straightforward application of the variation of constants formula yields the solution

$$
\psi_1(\tau) = -\left(\frac{1}{2} \sin(\tau) + \omega_1 \cos(\tau)\right) \tau.
$$

The parenthesized term is a linear combination of sines and cosines with the same frequency, and hence it is a sinusoidal function itself. In particular,

$$
\frac{1}{2} \sin(\tau) + \omega_1 \cos(\tau) = A \sin(\tau + \theta), \quad \text{with } A = \sqrt{\frac{1}{4} + \omega_1^2} \quad \text{and} \quad \theta = \arctan(2\omega_1),
$$

and hence, finally,

$$
\psi_1(\tau) = -A \tau \sin(\tau + \theta).
$$

In other words, $\psi_1$ is comprised of a sole, secular term. To suppress secularities, one would need to set $A = 0$; nevertheless, that is impossible as $A = \sqrt{\frac{1}{4} + \omega_1^2}$. [In fact, this formula suggests that the choice $\omega_1 = i/2$ will make $A$ vanish. You should research this possibility, in particular as far as higher order terms in the expansion(s) are concerned. Keep in mind that $\sin(i \varepsilon \tau/2) = -i \sinh(\varepsilon \tau)$ and that the damping contributes an exponential term to the exact solution; also, that the damping-induced frequency shift is $O(\varepsilon^2)$. These two observations point to the idea of using $\omega_1$ to deal with damping and the even coefficients $\omega_{2n}$ to deal with the frequency shift. Does this work out to all orders?] It appears, therefore, that the method of strained coordinates cannot handle this problem.

The method in question fails because the problem involves not one but two timescales: an $O(\varepsilon)$ one, $\tau_1 = \varepsilon t/2$ (associated with damping), and an $O(1)$ one, $\tau_2 = \sqrt{1 - \varepsilon^2/4t}$ (associated with oscillatory motion). Having chosen $\omega_0 = 1$, we have made the implicit choice to focus on the latter. In that regime, the exponent of the exponential term $e^{-\varepsilon \tau/2}$ contributed by damping to the (exact) solution will be perceived as $O(\varepsilon)$ and thus expanded asymptotically (here, as a McLaurin series). Recalling the McLaurin series of the exponential, we plainly see that we must anticipate the presence of secular terms in our asymptotic expansion. The most straightforward way to avoid such secularities would be to avoid expanding the exponential, and this is precisely what the method of multiple scales does.

The method works as follows: first, next to the timescale $t_1 = t$ suggested by the formulation of the problem, we introduce a second timescale $t_2 = \varepsilon^\alpha t$ (where the parameter $\alpha$ is free, for the moment). We will attempt to derive a solution to the problem of the form $y(t; \varepsilon) = \psi(t_1, t_2; \varepsilon)$. Since both $t_1$ and $t_2$ are defined in terms of the temporal variable $t$, chain rules yields

$$
\frac{d}{dt} = \frac{\partial}{\partial t_1} + \varepsilon^\alpha \frac{\partial}{\partial t_2} = \partial_1 + \varepsilon^\alpha \partial_2,
$$

where the last equation merely introduces notation. Our initial value problem becomes accordingly

$$
(\partial_1 + \varepsilon^\alpha \partial_2)^2 \psi + \varepsilon (\partial_1 + \varepsilon^\alpha \partial_2) \psi + \psi = 0, \quad \text{with } \psi(0, 0; \varepsilon) = 0 \quad \text{and} \quad (\partial_1 + \varepsilon^\alpha \partial_2) \psi(0, 0; \varepsilon) = 1.
$$

(5.7)

Note that this initial value problem involves a PDE and not an ODE; in other words, the notation we use above suggests that $t_1$ and $t_2$ are treated as variables independent of each other although they are both functions of the same variable $t$. Introducing, here also, the asymptotic expansion $\psi(t_1, t_2; \varepsilon) = \sum_{n \geq 0} \varepsilon^n \psi_n(t_1, t_2)$ and substituting in the initial value problem (5.7), we obtain at $O(1)$

$$
\partial_1^2 \psi_0 + \psi_0 = 0, \quad \text{with } \psi_0(0, 0) = 0 \quad \text{and} \quad \partial_1 \psi_0(0, 0) = 1.
$$

(5.8)
The solution to this linear, second order, parabolic PDE is readily found to be

\[ \psi_0(t_1, t_2) = C_+(t_2) \cos(t_1) + C_-(t_2) \sin(t_1), \quad \text{with} \quad C_+(0) = 0 \quad \text{and} \quad C_-(0) = 1. \tag{5.9} \]

Collecting the next order terms from each term entering the left member of (5.7), we obtain

\[ 2\varepsilon^\omega \partial_{1,2} \psi_0 + \varepsilon \partial_1^2 \psi_1 + \varepsilon \partial_1 \psi_0 + \varepsilon \psi_1 = 0, \quad \text{with} \quad \psi_1(0,0) = 0 \quad \text{and} \quad \varepsilon \partial_1 \psi_1(0,0) + \varepsilon^\alpha \partial_2 \psi_0(0,0) = 0. \tag{5.10} \]

If \( \alpha > 1 \), then the first term in the left member of the PDE above, as well as the second term in the left member of the second initial condition, are higher order. The remaining terms comprise a problem that is resonant because of the inhomogeneity, i.e., the advective term \( \partial_1 \psi_0 \). Hence, \( \alpha \leq 1 \). Next, if \( \alpha < 1 \), then the first term in the left member of the PDE is leading order (and similarly for the second term in the left member of the second initial condition); hence,

\[ \partial_1 \partial_2 \psi_0 = 0, \quad \text{with} \quad \psi_1(0,0) = 0 \quad \text{and} \quad \partial_2 \psi_0(0,0) = 0. \]

Substituting from (5.9) into this problem, we obtain

\[ -C'_+(t_2) \sin(t_1) + C'_-(t_2) \cos(t_1) = 0, \quad \text{with} \quad C'_+(0) = 0. \]

As discussed above, the left member of the first equation is a sinusoidal term with amplitude equal to \( \sqrt{[C'_+(t_2)]^2 + [C'_-(t_2)]^2} \); hence, it vanishes iff it holds identically that \( C'_+(t_2) = C'_-(t_2) = 0 \). Recalling the initial conditions for \( C_\pm \) reported in (5.9), we conclude that \( C_+(t_2) = 0 \) and \( C_-(t_2) = 1 \). Equivalently, \( \psi_0(t_1, t_2) = \sin(t_1) \), and resonance will once again appear in the next-order problem involving \( \psi_1 \). Hence, we select \( \alpha = 1 \) (as the principle of least degeneracy also dictates) to recast (5.10) in the form

\[ \partial_t^2 \psi_1 + \psi_1 = -2\partial_{1,2} \psi_0 - \partial_1 \psi_0, \quad \text{with} \quad \psi_1(0,0) = 0 \quad \text{and} \quad \partial_1 \psi_1(0,0) + \partial_2 \psi_0(0,0) = 0. \]

Substituting from (5.9) for \( \psi_0 \), we rewrite the PDE above in the form

\[ \partial_t^2 \psi_1 + \psi_1 = \left[ 2C'_+(t_2) + C_+(t_2) \right] \sin(t_1) - \left[ 2C'_-(t_2) + C_-(t_2) \right] \cos(t_1). \]

The only way to avoid resonance is to set the coefficients of the sinusoidal terms in the inhomogeneity equal to zero, \( 2C'_+(t_2) + C_+(t_2) = 0 \). Recalling also the initial conditions for \( C_\pm \) reported in (5.9), we conclude that \( C_+(t_2) = 0 \) and \( C_-(t_2) = e^{-t_2/2} \). Hence,

\[ \psi_0(t_1, t_2) = e^{-t_2/2} \sin(t_1). \]

Resubstituting for \( t_1 \) and \( t_2 \) from their definitions—\( t_1 = t \) and \( t_2 = \varepsilon t \) (recall that \( \alpha = 1 \))—we obtain the leading order term in the multiscale asymptotic expansion of \( y \),

\[ y_0(t) = \psi_0(t, \varepsilon t) = e^{-\varepsilon t/2} \sin(t). \]

This clearly matches the exact solution to leading order, where by ‘leading order’ we understand the process of expanding asymptotically the sinusoidal term while keeping the exponential intact. The same process can be repeated to all orders to yield higher order terms in the multiscale asymptotic expansion of the solution.
Homework set no.3

due March 12, 2013

01. Consider the various statements in Section 3.4.

(a) Prove statement #01, under the normal hyperbolicity assumption in that same section.

(b) Prove the second half of Statement #02; namely, that $T_z M^0$ and $T_z F^0(z)$ intersect transversally in $\mathbb{R}^{n+\ell}$.

(c) Prove formula (3.22) for $D\hat{G}_0(x, \hat{y})$.

02*. Consider the initial-value problem

$$\dot{y}(t) = (1 - y(t)) y^2(t), \quad \text{with } y(0) = \varepsilon.$$ 

Here, $0 < \varepsilon \ll 1$ as usual. Perform a full analysis of this model for $t \in \mathbb{R}_+$. In particular: locate the boundary layer, derive inner and outer expansions for the solution (to leading order, at least), and match them over properly chosen intermediate timescales. [Note that the approximate formula(s) for the outer solution(s) can—and therefore should!—yield the solution $y$ as an explicit function of $t$. The inner solution, on the other hand, can only be obtained implicitly.]

03. Consider the following initial-value problem for a harmonic oscillator with weak cubic damping,

$$\ddot{y}(t) + \varepsilon \dot{y}^3(t) + y(t) = 0, \quad \text{with } y(0) = 0 \text{ and } \dot{y}(0) = 1.$$ 

Use the method of multiple scales (two-timing) to derive a leading order solution which remains uniformly valid over the entire positive real axis.

04. Consider the following Mathieu equation describing a weakly parametrically forced harmonic oscillator,

$$\ddot{y}(t) + (\delta(\varepsilon) + \varepsilon \cos(\omega t)) y(t) = 0, \quad \text{with } y(0) = y_I \text{ and } \dot{y}(0) = 0.$$ 

Here, $\delta(\varepsilon)$ is assumed to possess the asymptotic expansion $1 + \sum_{n \geq 1} \varepsilon^n \delta_n$.

(a) Obtain the leading order behavior of the solution to the above initial-value problem for $\omega > 2$, by using a two-timescale expansion (in the original time variable $t$ and the slow time variable $\tau = \varepsilon t$).

(b*) Do the same for the case $\omega = 2$ and show that, if $|\delta(\varepsilon) - 1|$ is small enough, then the origin is destabilized. Is the origin destabilized for other values of $\omega$ as well?
6.1 Metastable patterns in a reaction–diffusion system

A summary of the theory of metastable patterns in a simple reaction–diffusion system is coming up; until then, your primary sources of information are Carr & Pego and Fusco & Hale.
References


[Holmes] M. Holmes, Introduction to Perturbation Methods, Springer-Verlag, New York, 1995 [available online in google books]


