01. Consider a 1-D wave equation on the real line with a time-invariant forcing term:
\[ u_{tt} - c^2 u_{xx} = P(x), \quad \text{for} \quad -\infty < x < \infty \quad \text{and} \quad t > 0, \]
\[ u(x,0) = \phi(x) \quad \text{and} \quad u_t(x,0) = \psi(x), \quad \text{for} \quad -\infty < x < \infty. \] (1) (2)

(a) Apply an Ansatz of the form
\[ u(x,t) = U(x,t) + f(x) \]
and choose the function \( f \) so that (1)–(2) is transformed into a homogeneous problem for \( U \).

(b) This problem can also be solved with the help of the Riemann function. Does one obtain the same result using that method?

02. Consider the partial differential equation
\[ u_t = ku_{xx} + au_x + bu \] (3)
with constants \( k, a, b \) and \( k > 0 \). Find constants \( \alpha(a,b) \) and \( \beta(a,b) \) such that the function \( w \) defined by
\[ w(x,t) = e^{\alpha(a,b)x + \beta(a,b)t} u(x,t) \]

satisfies the standard heat equation \( w_t = kw_{xx} \). (These constants will also depend on \( k \); we just suppress that dependence.) Also, show that
\[ \alpha(0,b) + \alpha(a,0) = \alpha(a,b) \quad \text{and} \quad \beta(0,b) + \beta(a,0) = \beta(a,b). \]

Try to use these results to interpret the effect that the zeroth- and first-order terms have on the function \( u \) determined by (3).

03. Consider a long, thin, homogeneous bar of length \( L \). Let \( u(x,t) \) be the temperature of the bar at position \( x \) and time \( t \). We assume that the surface of the bar is poorly insulated, so that thermal energy flows out through it. The local change in temperature due to loss of energy is proportional to the difference between the local temperature \( u \) of the bar and the temperature \( T \) of the surrounding medium (assumed to be constant). This leads to the following equation for the temperature distribution in the bar:
\[ u_t = ku_{xx} - A(u - T) \] (4)

with positive heat diffusion constant \( k \) and a positive transfer coefficient \( A \).

(a) Formulate an initial condition that models a temperature distribution according to a given function \( f \).

Also, formulate boundary conditions corresponding to insulated ends of the bar at \( x = 0 \) and \( x = L \).

(b) Transform the problem to an initial-boundary-value problem for the standard heat equation.

(c) Using the Fourier-series solution of the transformed equation, write the formal solution of the original problem for (4).

04. Let \( u \) satisfy the standard heat equation on \( [0, 1] \) with Dirichlet boundary conditions,
\[ u_t = u_{xx}, \quad \text{with} \quad u(0,t) = u(1,t) = 0 \quad \text{and} \quad u(x,0) = x(1-x). \]

Let \( T > 0 \). Calculate \( \max(u(x,t)) \) over the rectangular domain \([0, 1] \times [0, T]\) in the \((x,t)\)-plane.

Solutions are to be handed in at the practice session of January 13, 2012 (13:45 in CR 2H). Good luck!