01. In this exercise, we will derive a formula for the solution to Laplace’s equation in a half-plane,
\[ u_{xx} + u_{yy} = 0, \quad \text{with} \quad x \in \mathbb{R} \quad \text{and} \quad y > 0, \quad \text{and subject to the boundary condition} \quad u(x, 0) = f(x). \]

(a) Apply separation of variables to derive an eigendecomposition for the solution \( u \). Demand that \( u \) remains bounded to deduce that all negative numbers (and only these) are eigenvalues.

(b) Write \( u \) as a linear superposition of the eigenfunctions; since the spectrum is continuous, we write
\[
 u(x, y) = \int_0^\infty \left[ a_\omega \cos(\omega x) + b_\omega \sin(\omega x) \right] e^{-\omega y} d\omega. 
\]
(1)
Now use the boundary condition to express the coefficients \( a_\omega \) and \( b_\omega \) in terms of \( f \).

(c) Use the identity
\[
\int_0^\infty e^{-a\omega} \cos(b\omega) \, d\omega = \frac{a}{a^2 + b^2}
\]
and the formulas for \( a_\omega \) and \( b_\omega \) derived above to rewrite (1) in its final form,
\[
 u(x, y) = \frac{y}{\pi} \int_{-\infty}^\infty \frac{f(\xi)}{y^2 + (\xi - x)^2} \, d\xi.
\]

02. Assume that \( \Omega \) is a domain (in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \)) bounded by a piecewise smooth closed curve \( \partial \Omega \). Let \( u, v \in C^2(\bar{\Omega}) \).

(a) Prove Green’s first identity,
\[
\iint_{\Omega} (u \Delta v + \nabla u \cdot \nabla v) \, dA = \oint_{\partial \Omega} u \frac{\partial v}{\partial n} \, ds,
\]
(2)
where \( \partial v/\partial n = \hat{n} \cdot \nabla v \) is the directional derivative of \( v \) along \( \hat{n} \)—the outwards pointing, unit vector field normal to the boundary \( \partial \Omega \). Use this to derive Green’s second and third identities,
\[
\iint_{\Omega} (v \Delta u - u \Delta v) \, dA = \oint_{\partial \Omega} \left( v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) \, ds \quad \text{and} \quad \iint_{\Omega} \Delta u \, dA = \oint_{\partial \Omega} \frac{\partial u}{\partial n} \, ds,
\]
respectively.

(b) Suppose that \( u \) is harmonic in \( \Omega \), and that \( u \) satisfies homogeneous Dirichlet boundary conditions:
\[
\Delta u(x) = 0, \quad \text{for} \quad x \in \Omega, \quad \text{and} \quad u(x) = 0, \quad \text{for} \quad x \in \partial \Omega.
\]
Use (2) to show that \( u(x) = 0 \) for all \( x \in \bar{\Omega} \). Can you derive this same result in another way?

03. Prove the mean value property for harmonic functions in a bounded domain \( D \subset \mathbb{R}^2 \),
\[
 u(x) = \frac{1}{2\pi R} \oint_{\partial B(x, R)} u(y) \, ds(y),
\]
(3)
in two different ways. Here, the integral extends over the circle \( \partial B(x, R) \) which forms the boundary of a disk \( B(x, R) \) centered at \( x \in \Gamma \) and having a radius \( R \) sufficiently small to ensure that \( B(x, R) \subset D \).
(a) First, shift $x$ to the origin by a linear change of coordinates. Then, switch to polar coordinates and use Poisson’s formula,

$$u(r, \theta) = \frac{\rho^2 - r^2}{2\pi} \int_0^{2\pi} \frac{u(\rho, \phi)}{\rho^2 - 2r\rho \cos(\phi - \theta) + r^2} \, d\phi,$$

where $(r, \theta)$ is any point in the (shifted) domain.

Here, $\rho$ is any positive number such that $B(0, \rho)$ is fully contained in that domain.

(b) Use, instead, the representation theorem for (twice continuously differentiable) functions in $\mathbb{R}^2$ (cf. HW #04, problem #02),

$$u(x) = \frac{1}{2\pi} \oint_{\partial \Omega} \left[ \frac{\partial u(y)}{\partial n} \ln \frac{1}{|y - x|} - u(y) \frac{\partial}{\partial n} \ln \frac{1}{|y - x|} \right] \, ds(y) - \frac{1}{2\pi} \iint_{\Omega} \Delta u(y) \ln \frac{1}{|y - x|} \, dA(y).$$  \hspace{1cm} (4)

Here, $\Omega \subset D$ is any subdomain of $D$.

Good luck!