Notes on the 1–dimensional wave equation on the half line

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In these notes, we will treat the 1–D wave equation on the half line \((x \geq 0)\) and with a Dirichlet boundary condition (no displacement at the boundary):

\[
\begin{align*}
    u_{tt} - c^2 u_{xx} &= 0, & \text{for } x > 0 \text{ and } t > 0, \\
    u(x, 0) &= \phi(x) \quad \text{and} \quad u_t(x, 0) = \psi(x), & \text{for } x > 0, \\
    u(0, t) &= 0, & \text{for } t \geq 0.
\end{align*}
\]

As we discussed briefly in class, the idea is to turn this problem into a problem on the entire real line by extending the initial conditions to the region \(x < 0\) (method of images). Then, the solution to our problem can be obtained by restricting d’Alembert’s solution for this extended problem back to the region \(x > 0\). Since d’Alembert’s solution does not admit a fixed value at \(x = 0\) (i.e., we cannot impose the boundary condition on it), it follows that one must select the extension carefully so as to automatically satisfy the condition \(u(0, t) = 0\).

To achieve this, we extend \(\phi\) and \(\psi\) in an odd fashion:

\[
\Phi(x) = \begin{cases} 
\phi(x), & \text{for } x > 0, \\
-\phi(-x), & \text{for } x < 0
\end{cases}, \quad \text{and} \quad \Psi(x) = \begin{cases} 
\psi(x), & \text{for } x > 0, \\
-\psi(-x), & \text{for } x < 0
\end{cases}
\]

The rationale behind this is that displacements and velocity “packets” (see the ‘prerequisites’ set of notes) arrive at \(x = 0\) from the points \(x > 0\) and \(-x < 0\) at the same time instant, \(t = x/c\). The simplest choice that lets them annihilate each other, thus leaving the origin unaffected (i.e., at rest), is to extend the initial conditions as in (2), as then \(\Phi(x) + \Phi(-x) = 0\) and \(\Psi(x) + \Psi(-x) = 0\).

More explicitly, d’Alembert’s formula yields

\[
U(x, t) = \frac{1}{2} \left[ \Phi(x - ct) + \Phi(x + ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \Psi(\chi) \, d\chi,
\]

for \(-\infty < x < \infty\) and \(t > 0\). Here, \(U\) is the solution to the extended problem

\[
\begin{align*}
    U_{tt} - c^2 U_{xx} &= 0, & \text{for } -\infty < x < \infty \text{ and } t > 0, \\
    U(x, 0) &= \Phi(x) \quad \text{and} \quad U_t(x, 0) = \psi(x), & \text{for } -\infty < x < \infty.
\end{align*}
\]

We claim that \(u\) is the restriction of \(U\) on the domain \(x > 0\) and \(t > 0\). First, \(U\) satisfies the wave equation for all \(x > 0\) and \(t > 0\) as it also satisfies it for (the superset) \(-\infty < x < \infty \text{ and } t > 0\). Hence, the first equation (PDE) in (1) is satisfied. Next, the second equation (initial conditions) in (1) is also satisfied by virtue of our definition (2). Finally, the boundary condition (2) is also satisfied by means of the calculation (cf. (3))

\[
U(0, t) = \frac{1}{2} \left[ \Phi(-ct) + \Phi(ct) \right] + \frac{1}{2c} \int_{-ct}^{ct} \Psi(\chi, t) \, d\chi
\]
\[ u(x, t) = \frac{1}{2} [-\Phi(ct) + \Phi(ct)] + \frac{1}{2c} \left[ -\int_{-ct}^{0} \Psi(-\chi) d\chi + \int_{0}^{ct} \Psi(\chi) d\chi \right] = 0. \]

Hence, the restriction of \( U \) for positive values of \( x \) satisfies (1) and thus solves our original problem as advertised.

It is interesting to examine the behavior of \( U \) in the region of interest, \( x > 0 \). We distinguish two cases, \( ct \leq x < \infty \) and \( 0 \leq x < ct \) (note that these correspond to two mutually disjoint and complementary regions of the domain \([0, \infty)^2\) of the \((x, t)\)-plane).

**Case** \( ct \leq x < \infty \). In that case, both arguments of \( \Phi \) in (3) and both bounds of the integral in the same equation are positive. Hence, (3) becomes

\[ u(x, t) = \frac{1}{2} [\phi(x - ct) + \phi(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(\chi) d\chi, \tag{4} \]

which is d’Alembert’s formula for the wave equation on the entire real line. This is, indeed, only to be expected, as all points \((x, t)\) in that region are inaccessible by the ‘mirror image’ of the initial conditions in the negative axis (since any point on the negative axis needs an amount of time at least as large as \( x/c > t \) to reach \((x, t)\)).

**Case** \( 0 \leq x < ct \). In that case, \( x - ct < 0 < x + ct \) in (3), and hence this equation becomes

\[ u(x, t) = \frac{1}{2} [-\phi(ct - x) + \phi(x + ct)] + \frac{1}{2c} \int_{ct-x}^{ct+x} \psi(\chi) d\chi, \tag{5} \]

where we have used that \( \int_{x-ct}^{ct-x} \psi(\chi) d\chi = 0 \), since \( \Psi \) is odd and the integration interval is symmetric about zero. Note that an interpretation of this formula is the following: at time zero and for every \( x \), the initial displacement profile splits in two and each half starts traveling in opposite directions. When the half that moves to the left reaches the origin, it is reflected (i.e., it reverses the direction it travels in) and its amplitude reversed. This half keep traveling to the right, then, with the amplitude reversed. (The exact same picture applies to the velocity profile, too.)