The final assignment can be made individually or in pairs.

Assignment

Define the quadratic map

\[ f(x) = r - x^2. \]

1. Use Mathematica to draw an orbit diagram with \( r <= 2 \).

2. Find the fixed points of \( f \). Determine for which values of \( r \) the fixed points exist, and for which values of \( r \) they are stable or unstable. Draw a bifurcation diagram where unstable points are plotted with dashed lines. Combine the bifurcation diagram with the orbit diagram.

Bifurcation curves

Points of order \( n \) with \( n > 1 \) can be found by finding the zeros of the function \( f^n(x) - x \). In order to exclude points of order \( k \) where \( k \) is a divisor of \( n \), it is better to find the zeros of \( (f^n(x) - x)/(f^k(x) - x) \).

3. Find the points of order 2. Draw the points of order 2 in the combined orbit/bifurcation diagram.

4. Do the same thing for points of order 4 and 3.

Critical polynomials

The critical polynomials \( p_n(r) \) of \( f \) are defined as

\[ p_n(r) = f^n(0). \]

(In general: \( p_n(r) = f^n(x_0) \) with \( f'(x_0) = 0 \).)

5. For \( n = 1, 2, 3, 4, 5 \), draw the critical polynomials \( p_n(r) \) in the orbit/bifurcation diagram.

6. In the ‘period-doubling range’ the critical polynomials seem to approach the points in the orbit diagram. Can you explain this?

The Feigenbaum constant

The points of order \( 2^n \) can be found as the zeros of \( (f^{2^n}(x) - x)/(f^{2^{n-1}}(x) - x) \). Define \( r_n \) as the smallest value of \( r \) for which the orbits of period \( 2^n \) occur.

7. Find \( r_1 \) and \( r_2 \).

8. With the approximation \( \delta = 4.669 \) for the first Feigenbaum constant, use the result of (7) to find an estimate for the smallest value of \( r \) for which chaos exists.
The Lyapunov exponent

The Lyapunov exponent can be used to detect the values of $r$ for which there are chaotic orbits, see section 10.5 of Strogatz’s book.

(9) Study example 10.5.3 in Strogatz’s book, and draw a graph similar to fig. 10.5.2.

(10) For which value of $r$ chaos emerges for the first time (approximately)?