Introduction to Mathematics and Modeling

Midterm Test 1

Motivate all your answers.
The use of electronic devices is not allowed.

1.

Given is a circle with radius 3 and center O. The angle $\angle COB$ is $60^\circ$.

(a) [1 pt] Express angle $\theta = \angle AOB$ in radians.

(b) [1 pt] Find the length of the arc from $A$ to $B$.

2. [1 pt] Of two angles $\alpha$ and $\beta$ the sine and cosine values are given:

<table>
<thead>
<tr>
<th></th>
<th>$\cos$</th>
<th>$\sin$</th>
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</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$\frac{1}{2}\sqrt{3}$</td>
<td>$-\frac{1}{2}$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$\frac{1}{2}$</td>
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Find $\sin(\alpha + \beta)$.

3. [2 pt] The function $f$ is defined by

$$f(x) = \frac{1}{1 - \frac{1}{x + 1}}$$

for all $x \neq 0$.

Prove that $f$ is one-to-one.
4. (a) [2 pt] Evaluate
\[ \log_3(9) + \frac{\ln \left( \frac{1}{2} \right)}{\ln(2)}. \]

(b) [1 pt] Use the approximations \( \ln 2 = 0.7 \) and \( \ln 10 = \frac{7}{3} \) to calculate an approximation of \( \log_2 1000 \).

**Note:**
Simplify your answers as much as possible. The final results should not contain any logarithms.

5. [2 pt] An exponentially decaying quantity \( y \) satisfies the following function:
\[ y(t) = 100 \cdot (0.1)^t, \]
where \( t \) is measured in seconds. How many seconds does it take for \( y \) to decay to 1% of the initial quantity (the quantity at time \( t = 0 \))? Put in other words: find \( T \) such that \( y(T) \) is 1% of \( y(0) \).

**Total:** 10 points

Note: there is a trigonometric formula sheet attached to this test.
Trigonometry formulas

1. \( \cos(-\alpha) = \cos \alpha \)
2. \( \sin(-\alpha) = -\sin \alpha \)
3. \( \cos^2 \alpha + \sin^2 \alpha = 1 \)
4. \( \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \)
5. \( \sin(\alpha + \beta) = \cos \alpha \sin \beta + \sin \alpha \cos \beta \)
6. \( \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \)
7. \( \sin(\alpha - \beta) = \cos \alpha \sin \beta - \sin \alpha \cos \beta \)
8. \( \cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha \)
9. \( \sin(2\alpha) = 2 \sin \alpha \cos \alpha \)
10. \( \sin \alpha \cos \alpha = \frac{1}{2} \sin(2\alpha) \)
11. \( \cos^2 \alpha = \frac{1}{2} + \frac{1}{2} \cos(2\alpha) \)
12. \( \sin^2 \alpha = \frac{1}{2} - \frac{1}{2} \cos(2\alpha) \)
13. \( \sin \alpha \cos \beta = \frac{1}{2} \sin(\alpha - \beta) + \frac{1}{2} \sin(\alpha + \beta) \)
14. \( \cos \alpha \cos \beta = \frac{1}{2} \cos(\alpha - \beta) + \frac{1}{2} \cos(\alpha + \beta) \)
15. \( \sin \alpha \sin \beta = \frac{1}{2} \cos(\alpha - \beta) - \frac{1}{2} \cos(\alpha + \beta) \)
16. \( \tan \alpha = \frac{\sin \alpha}{\cos \alpha} \)
17. \( \tan(-\alpha) = -\tan \alpha \)

Special angles

<table>
<thead>
<tr>
<th>( \alpha )</th>
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<th>( \sin \alpha )</th>
<th>( \tan \alpha )</th>
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