English Summary of the Report in Dutch
An Analysis of the "Memory" Game, 65-Afternoon Project Report, University of Twente, Department of Electrical Engineering, 1983.

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1 Context

The summary below is actually a letter that I addressed to Uri Zwick in 1992. I had contacted him after reading about his work in the *Scientific American* [1] and telling him about a student project that I had performed back in 1983 under the supervision of Dr. Frits Göbel. We had independently discovered similar results as mentioned in the Scientific American.

Uri Zwick sent me a draft of [2]. I reacted by sending him my student-project report (handwritten, in Dutch) and a summary in English. That reaction is reproduced below. The correspondence resulted in the report being mentioned in the final version of [2]. The report in Dutch is available through my website `http://utelnt.el.utwente.nl/links/gerez/`.

2 Letter

Dear Uri,

Below I will present a summary in English of my 65-afternoon project report and I will point out some relations with the material presented in your paper "The Memory Game".

- Chapter 1 gives is an introduction: it presents the project description and the structure of the report.
- Chapter 2 describes the memory game as it will be analyzed, i.e. with perfect memory.
- Chapter 3 begins with an introduction to the notion of dynamic programming (Section 3.1).
- Next follows the definition of the "state vector" (Section 3.2.1), which is defined as a triplet $\langle K, B, P\rangle$. Here K represents the number of cards present in the game (always an even number,

it corresponds to $2n$ in your notation), B represents the number of cards whose identity is known (k in your notation) and P is the number of cards whose identity is known together with the matching card (they can be picked up by the player that has the turn; you don't consider these configurations, but I wanted to start with a formulation that is as general as possible). The inequality at the end of Page 8 states that there is an upper bound for B that is the average of K and P.

- Section 3.2.2 introduces the gain function, denoted by $w(K, B, P)$. As opposed to your function $e_{n,k}$, this function is always positive and simply gives the expected number of cards that a player will have when continuing the game from the configuration $\langle K, B, P \rangle$. The general expression presented just below the middle of Page 9 simply states that the expected gain is the sum of the expected gains reached from the states after picking up two cards weighted by the probabilities to reach them and those that will make it necessary to switch turns with the other player (again weighted by the correct probability). Using our gain function, one has to take the function's complement with respect to K when losing the turn. Using the notation $w_i(K, B, P)$ for the expected gain using move i, in Page 10 it is stated that the expected gain in a certain state is the maximum of the w_i's, while the i that corresponds with the maximum, is the best move.

- In Section 3.3.1 an attempt is made to classify all possible moves ("tactics"), not excluding those that might not look clever. On Page 11 the sets K_1 and K_2, representing the possible moves for Ply 1 and Ply 2 respectively, are introduced. K_1 consists of: "pick up a card that has a known matching card", "pick a known card without matching card" and "pick an unknown card". K_2 contains these plus the move "pick up the known card that matches the card of Ply 1". There two categories of moves: *unconditional* moves in which the decision for Ply 2 is independent of the result of Ply 1 and *conditional* moves in which the decision is dependent.

- Section 3.3.2 shows how to compute the expected gain of some example moves. Section 3.3.3 introduces all possible moves, preceded by a table that defines the probabilities p_1 through p_9, which are used in the expressions for the expected gain of the different moves. Besides, for each move, the states in which it can be applied, are specified. I will only explain the moves that are relevant for the rest of the report.

  - T_1 collects a pair of matching cards.
  - T_2 picks up a pair of known nonmatching cards and amounts to a "pass".
  - T_4 picks up an unknown card together with a known card (the order does not matter as this is an unconditional move).
  - T_5 picks up an unknown card and the matching card if there is one; it picks up a known card in Ply 2 otherwise. This corresponds to your "1-move".
  - T_6 picks up an unknown card and the matching card if there is one; it picks up an unknown card in Ply 2 otherwise. This corresponds to your "2-move".
  - T_7 picks up two unknown cards.
  - T_8 starts with an unknown card; if it matches with another card, a known but *nonmatching* card is played in Ply 2; an unknown card is played otherwise.

- Sections 3.4 deals with the order in which the states have to be traversed to compute the optimal moves and expected gains. Section 3.5 gives an expression for the number of different states for $K \leq k_{max}$.

- Section 3.6 presents the results of the computer simulations. Two different variations on the game have been considered. In the first variant T_2 can be applied (when there are two known nonmatching cards); in the second variant T_2 has been disallowed (one can imagine that an extra rule states that the game can only finish when there are no cards left to be picked up, and that move T_2 is therefore not allowed). It turned out that move T_1 is always optimal when $P > 0$. Therefore only the optimal moves for $P = 0$ are presented. They are displayed on Pages 23 and 24 respectively for the two variants. The first table is essentially the same as Table 2 in your paper, with the exception that the anomaly for $n = 6, k = 1$ is absent. Is this a mistake I made during copying the results into the report? Perhaps I can find home the computer outputs of those days and check it. The second table gives a remarkable result: T_8 which leaves a matched pair for the opponent appears as the winning move in some states (a move comparable to a sacrifice in chess?). Note also that there are many anomalies for small values of K and B.

- Section 3.7 tries to supply some theoretical foundations for the results. In the second part of Section 3.7.1 a proof is given for the fact that T_1 is always optimal in the cases that it can be applied. It assumes that T_x is better than T_1 in a certain state and then derives a contradiction by showing that applying first T_1 and then T_x never gives a smaller expected gain. A problem arises only with T_2 as it cannot always be interchanged with T_1 (see the example mentioned at the end of Page 29): this gives rise to consider a variant of the game where "passing" is allowed in all states, as you also mention in Section 5 of your paper.

- Section 3.7.2 investigates the diagonals of the tables. A new variable N is introduced; each value of N selects another diagonal: $B = \frac{K}{2} - N$ or $N = \frac{K}{2} - B$. The analysis of $N = 0$ is trivial. For $N = 1$ the expected gain for the empirically found optimal move T_6, $w_6(K, \frac{K}{2} - 1, 0)$, is given a closer look and the difference equation that results is solved. The result is shown at the beginning of Page 32: $\frac{K+4}{3}$. It is then shown that for $K = 8$, T_2 gives the same gain and becomes superior for higher values of K. It is also shown that no other move can be superior to T_2 for larger K by induction (only the comparison with T_5 is elaborated). For the case that passing is not allowed, similar computations are performed and the transition to T_8 is proved.

- In Section 3.7.3 an attempt is made to prove that T_5 is superior to T_4. It turns out that the proof would be easy if passing was allowed unconditionally. This is another reason for giving special attention to the game with unconditional passing in Chapter 4.

- Section 4.1 eliminates most of the moves, viz. T_4, T_3, T_9, T_8 and T_7, by proving that either T_5 or T_6 is better. The application areas of T_5 and T_6 have to be enlarged to justify the elimination of some of those moves.

- Section 4.2 presents the results of the new computer simulations. Note that the table on Page 41 is exactly the same as your Table 5 and the one of Page 42 is the same as your Table 4 (you need to read the tables "mirrored" as the parameter N is used in my tables, and you should add n to the gain in your table).

- Section 4.3 tries to prove the results theoretically. In Section 4.3.1 an exact expression for w_5 is derived for $N = 2$. It is also mentioned that it is not difficult to see that T_5 will always be the better than T_2 (coefficient of K) and T_6 (induction proof).

- In Section 4.3.2 a so-called "coefficients of the highest power calculus" is introduced. It mainly lists in pairs the exponents of the variable together with its coefficients in decreasing order of the exponent. It will make it possible to approximate the behavior of the expected gain of moves by only considering the first two terms of the solutions.

- This calculus is applied to different values of N in Section 4.3.3. The last part of this section is the most interesting, as expressions for the expected gain are derived for general values of N, when N is even and odd respectively. When N is even, T_5 is the optimal move. Assuming that T_2 is the optimal move in the preceding column (corresponding to $N-1$), and using Theorem A.2, the following approximation is found on Page 52:

$$w_5(K, \frac{K}{2} - N) \simeq \frac{(N+1)K - N}{2N+1}$$

  I have checked that this is the same expression as the first one given in Theorem 3.2 of your paper. For N is odd, two assumptions are made: that T_2 is the optimal move two columns before $(N-2)$ and that the result found for w_5 holds for the previous column $(N-1)$. The expression found for w_6 predicts the transitions between T_6 and T_2. As stated on Page 54, $K = 6N + 2$ gives the site of the transition. This corresponds to your expression $k \leq \frac{2n+1}{3}$ given in Theorem 3.2 of your paper.

- Section 4.3.4 gives the problems that could not be proved in the span of my project: the alternation of T_5 and T_6 outside the transition regions with T_2, the behavior for $B = 0$, etc.

- Chapter 5 mentions some variations on the game: the variant with finite memory (with the memoryless game as an extreme case), a version where the cards that have been seen are put aside (in a bag for example), having n plies instead of 2, and having more than 2 players.

I hope that this helps you.

Best wishes, Sabih

References

[1] I. Stewart. Concentration: A winning strategy. *Scientific American*, 265(4):103–105, October 1991.

[2] U. Zwick and M.S. Paterson. The memory game. *Theoretical Computer Science*, 110:169–196, 1993.