Solution of problem 2003/4-B by Bert Jagers

We consider an obvious generalization. Let G be a group and m and n be coprime integers. Assume that all mth powers of elements of G commute as well as all nth powers. Then G is commutative.

Proof: Let M be the subgroup generated by all mth powers of elements of G and similarly N for the nth powers. Then M and N are commutative by our assumption. Moreover, M and N are clearly invariant under all automorphisms of G, and so, in particular, under all internal automorphisms. That is, M and N are normal subgroups of G. On the other hand, since GCD(m, n) = 1, there exist integers p and q such that

(1) $$1 = pm + qn.$$

Thus any element x of G can be written as a product of an element of M and N, namely

(2) $$x = (x^p)^m \, (x^q)^n.$$

In short, $G = M\,N$. Now let s and t be two arbitrary elements of M and N, respectively. Then it suffices to show that $s\,t = t\,s$ to settle the overall commutativity of G. It follows from (2) that the intersection of M and N is a subgroup of the center of G. Let $[s, t] = s\,t\,s^{(-1)}\,t^{(-1)}$, be the commutator of s and t. Then $[s, t]$ is the product of two elements of N (because N is normal) *but also* the product of two elements of M (because M is normal), hence $[s, t]$ lies in the center of G, $[s, t] = z$, say ,where $z\,x = x\,z$ for all elements x of G. Thus $s\,t\,s^{(-1)} = z\,t$ and $t\,s^{(-1)}\,t^{(-1)} = z\,s^{(-1)}$ and hence, since z is central, $s\,t^m\,s^{(-1)} = z^m\,t^m$ and $t\,s^{(-n)}\,t^{(-1)} = z^n\,s^{(-n)}$. But s and t^m commute (by our assumption) and so do t and $s^{(-n)}$. It follows that $z^m = 1$ *and* $z^n = 1$. Hence $z = (z^m)^p \, (z^n)^q$ is equal to 1 as well. In other words $s\,t = t\,s$, as required.

Remark: There is a slightly different route via the factor group $G\_ = G$ mod Z(G), where Z(G) is the center of G. Within $G\_$ the cosets $s\_ = s$ Z(G) and $t\_ = t$ Z(G) generate a commutative subgroup A, for $[s, t] = z$ is an element of Z(G). Since the orders of $s\_$ and $t\_$ are coprime in $G\_$, the subgroup A is actually cyclic (with generator the product $s\_\,t\_$, for instance). But any group which is cyclic modulo its center is trivially commutative.