Wave Reflection over Slowly Varying Bathymetry modeled by Effective Boundary Conditions

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Introduction

Fig. 1. Tsunami wave slams into the Aceh coast.

In tsunami simulations, the waveheight near the shore is the most important aspect scientists would like to calculate correctly. Unfortunately, the present-day simulation tools still cannot calculate the waveheight near the shore accurately enough. One source of inaccuracy is the interaction of the incoming waves with reflected waves from the coast. Besides, computing the details of run-up and run-down of waves on the shore is computationally very demanding and expensive since closer to the shore a finer computational resolution will be needed. Moreover, the modeling of the physical processes is bound to be rather rudimentary because many aspects of tsunami propagation, e.g. nonlinearity, dispersion, friction, etc. have to be considered.

Most of the tsunami simulations nowadays will use a fixed wall as boundary condition at the shore to simplify the problem. But this will cause inaccuracies in the reflected waves since in fact there are run-up and run down waves on the shore. Therefore, we will need to design boundary conditions that are able to calculate more accurate wave interactions near the shore without increasing the computational cost. Here, we will derive the so-called Effective Boundary Conditions (EBCs) to be imposed at some positions before the shoreline.

Effective Boundary Conditions

These EBCs are of general relevance and can be implemented in any numerical program to approximate the onshore tsunami flow without the necessity to calculate the detailed flooding and drying flows. The effect of the run-up of the waves on the shore when they return and interact after run-down with the incoming waves from the sea should be modeled in an approximate way in these EBCs. The illustration of the EBCs problem can be seen in fig. (2).

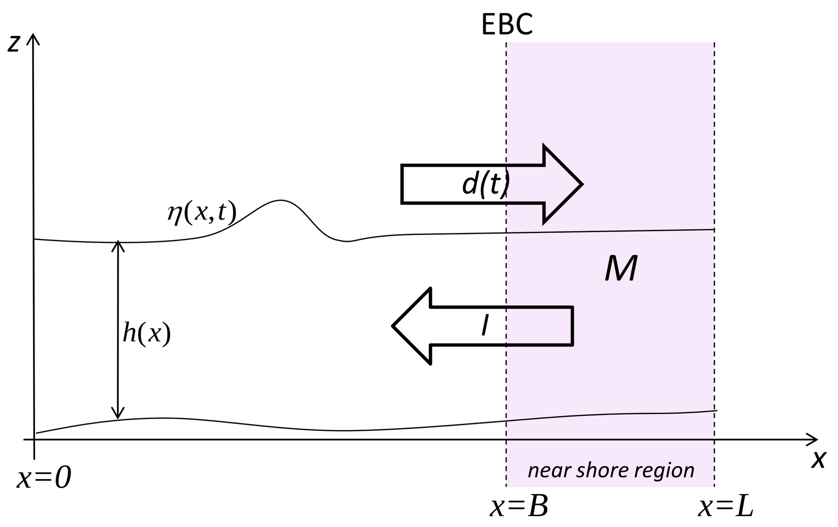

Figure 2. Effective boundary condition illustration.

The basic idea is that in a zone before the shoreline (at a position of given, nonzero depth), $x=B$, information of the incoming wave is 'measured' in time, without disturbing the waves. Denote this information symbolically by *d(t)*. Then a theoretical model is used to obtain the wave reflection by the run-up and run-down at near shore region *[B,L]*, and select the information that accounts for the reflected waves influx I into the sea at point $x=B$ before the shoreline. Symbolically this theoretical model can be denoted by *M(d)*, where at time t the result will depend on the incoming wave for all previous time, i.e.

$$I = M(d)(t) \tag{1}$$

depends on $d(\tau)$ for all $\tau<t$.

Linear Shallow Water Equations

The wave propagation model is based on Linear Shallow Water Equations (SWE), i.e.:

$$\partial_t \eta = -\nabla\cdot(h\nabla\phi) \tag{2a}$$

$$\partial_t \phi = -g\eta \tag{2b}$$

that can be rewritten as one second order in time equation for η and for ϕ as:

$$\partial_{tt}\eta - \nabla\cdot\left(c^2\nabla\eta\right)=0 \quad \text{or} \quad \partial_{tt}\phi - \nabla\cdot\left(c^2\nabla\phi\right)=0 \tag{3}$$

for $c=\sqrt{gh}$, η is the wave elevation, $u=\nabla\phi$ is the wave velocity, and h is the bathymetry profile.

For the numerical solution, we will use a Finite Element Method (FEM) with quadrilateral elements in two-dimensional. To check the EBC, we will compare the results with the simulation on the whole domain in a one-dimensional setting.

Influx Transparent Boundary Condition

For a right Influx Transparent Boundary Condition (ITBC), i.e. transparent for waves propagating to the right but influxing the waves propagating to the left, we assumed that the solution to the exterior of our computational domain is known. It is assumed that the solution outside the right BC is the propagating wave over a constant depth h_R at the right boundary. The right ITBC for Linear SWE is

$$\partial_t\phi + c_R\partial_x\phi = 2G' \tag{4}$$

with $c_R=\sqrt{gh_R}$. This is the condition for the right ITBC, where the influx signal at $x=x_R$ is given by $2G'$.

WKB Approximation

For waves above a slowly varying bathymetry, there is a very good approximation termed the Wentzel-Kramer-Brillouin (WKB) approximation [1],[2].
The variations in the water depth are slow if $\varepsilon = \Delta h/\bar{L} \ll 1$, where Δh is the change in depth over a horizontal distance $\bar{L}$ (fig. 3).

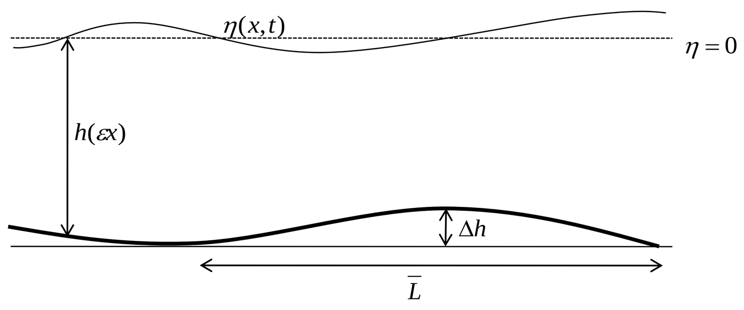

Fig. 3. Slowly varying bathymetry.

Following [1], the WKB approximation for right traveling waves is

$$\eta(x,t) = \frac{A}{\sqrt{c(\varepsilon x)}} F\big(\theta(x,t)\big) \tag{5}$$

where θ satisfies the eiconal equation

$$\left(\partial_t\theta\right)^2 = c^2\left(\partial_x\theta\right)^2. \tag{6}$$

It can be observed that this is precisely the solution of

$$\partial_t\eta + \sqrt{c}\,\partial_x\sqrt{c}\,\eta = 0. \tag{7}$$

which can be checked by substituting (5) to (7).

Reflection WKB Approximation

Because our aim is to look for the expression of reflected waves that travels to the left for right traveling waves, we will write the total wave elevation (η) as the sum of a right ($\vec{\eta}$) and a left (reflected) wave ($\overleftarrow{\xi}$), i.e.

$$\eta = \vec{\eta} + \overleftarrow{\xi} \tag{8}$$

The wave equation in one dimensional can be rewritten as

$$(\partial_t^2 - \mathcal{D}^2)\vec{\eta} + (\partial_t^2 - \mathcal{D}^2)\overleftarrow{\xi} = b\vec{\eta} + b\overleftarrow{\xi} \tag{9}$$

with the operator $\mathcal{D}(.) = \sqrt{c}(\partial_x\sqrt{c}(.))$ and $b(.) = \partial_x(c^2\partial_x(.)) - \sqrt{c}\partial_x(c\partial_x(\sqrt{c}(.)))$.

The first term in the left hand side vanishes by (7). The second term $b\overleftarrow{\xi}$ will be of higher order and will be neglected in the following. By defining $(\partial_t - \mathcal{D})\overleftarrow{\xi} = \lambda$, we can rewrite (8) and (10) as a system of differential equations

$$\begin{cases} (\partial_t + \mathcal{D})\vec{\eta} = 0 \\ (\partial_t + \mathcal{D})\lambda = b(x)\vec{\eta} \\ (\partial_t - \mathcal{D})\overleftarrow{\xi} = \lambda. \end{cases} \tag{10}$$

By doing some algebraic steps and transformation to the time independent variable $y=y(x)$ such that $\partial_y = c\partial_x \Rightarrow y = \int_0^x \frac{d\zeta}{c(\zeta)}$, given the initial condition $\eta(x,0)=F(x)$, then we have the solutions as:

$$\bar{\eta} = \bar{F}(y-t) \tag{11a}$$

$$\bar{\lambda} = \bar{F}(y-t)\mathrm{B}(y) \quad \text{with} \quad \mathrm{B}(y) = \int_0^y b(\zeta)d\zeta \tag{11b}$$

$$\bar{\xi} = \int_y^{y+t} \bar{F}(2\beta-(y+t))\mathrm{B}(\beta)d\beta \tag{11c}$$

for $\bar{\eta} = \sqrt{c}\vec{\eta}$, $\bar{\lambda} = \sqrt{c}\lambda$, $\bar{\xi} = \sqrt{c}\overleftarrow{\xi}$, and $\bar{F} = \sqrt{c}F$.

The comparison between the analytical solution of Reflection WKB approximation with the numerical solution is shown in fig. (4)-(5). The total simulation time is 25 minutes.

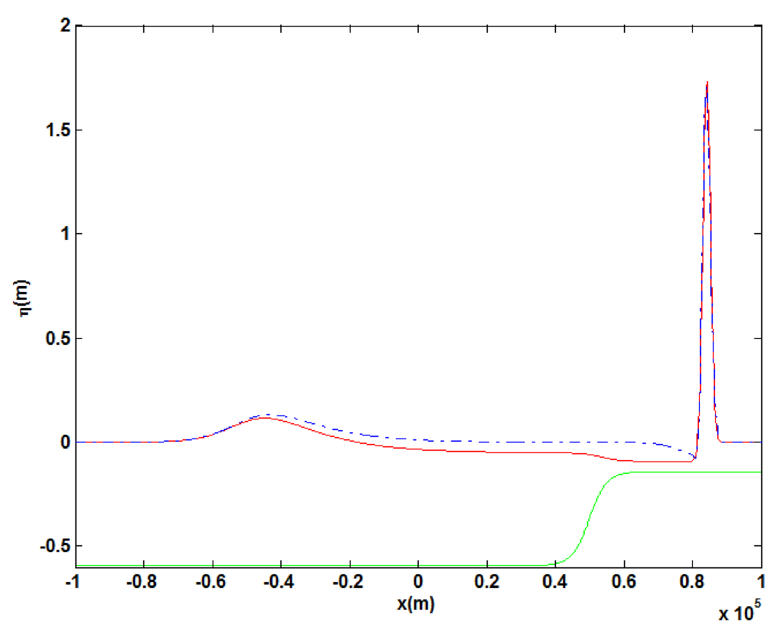

Fig. 4. Comparison between the numerical (blue dashed line) and the analytical (red solid line) solution for steep slope (1/15) . The green line shows the bathymetry.

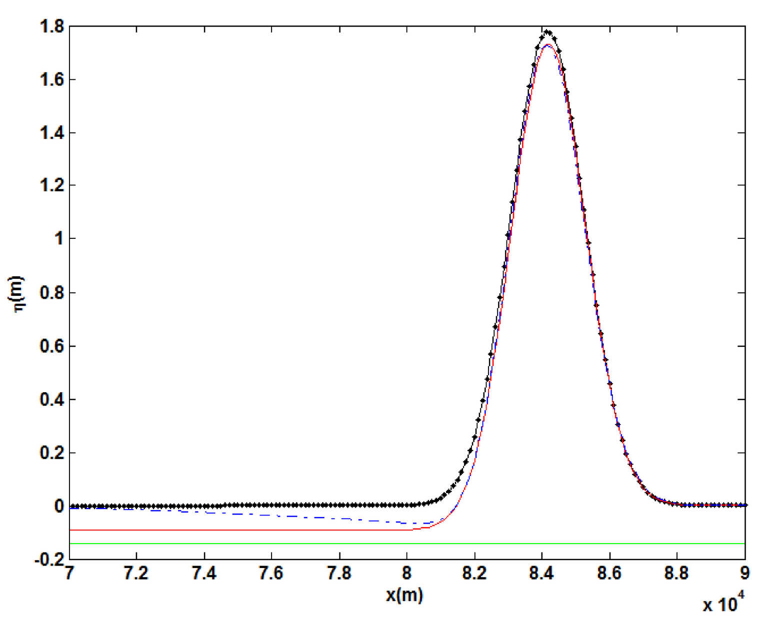

Fig. 5. Comparison between the WKB approximation $\vec{\eta}$ (black dotted line), the reflection WKB approximation $\vec{\eta}+\overleftarrow{\xi}$ (red solid line), and the numerical solution (blue dashed line) for steep slope (1/15). The green line shows the bathymetry.

Effective Boundary Condition over Slowly Varying Bathymetry

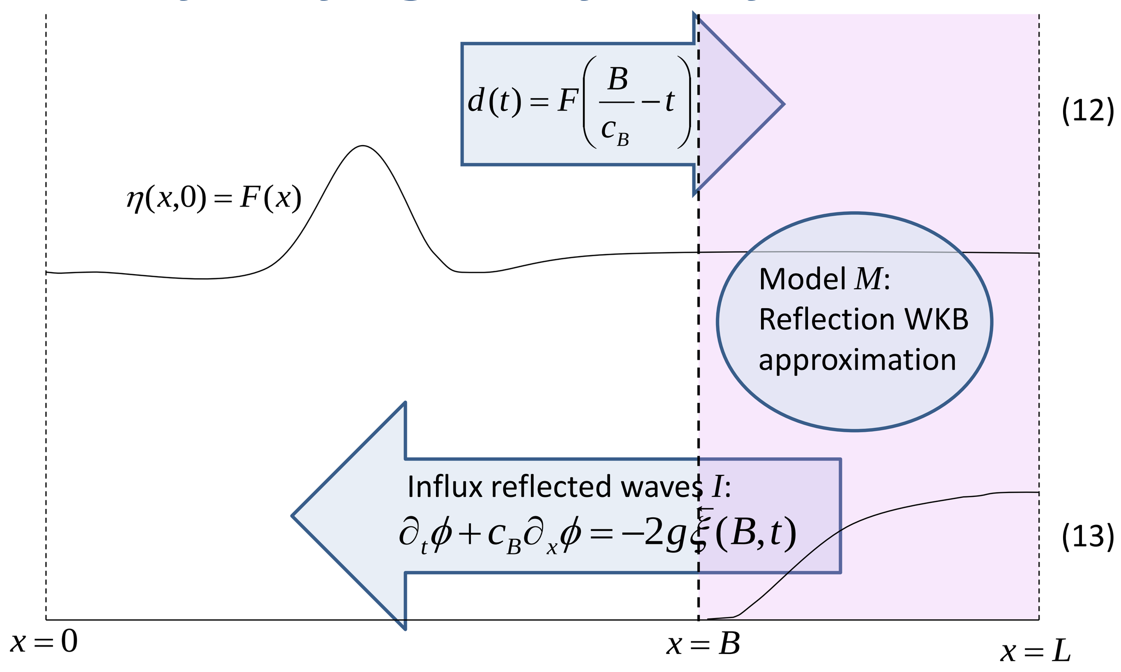

Fig. 6. Effective Boundary Condition over slowly varying bathymetry

The EBC over slowly varying bathymetry is illustrated in fig. (6). Domain [0,B] is solved numerically together with delayed boundary condition (13) at x=B.
For checking the EBC implementation in the domain [-100,20]km, we will compare it with the numerical solution in the whole domain [-100,100]km. The results are shown in fig. (7)-(9).

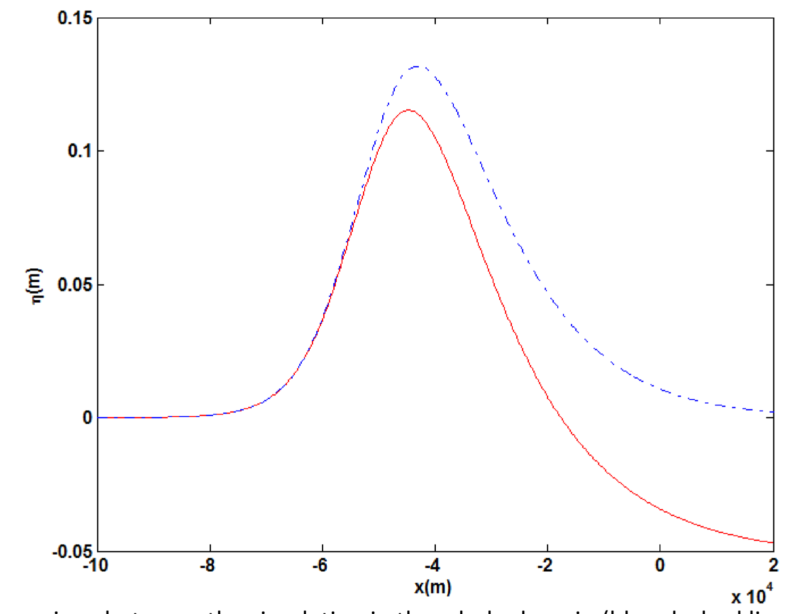

Fig. 7. Comparison between the simulation in the whole domain (blue dashed line) and using EBC (red solid line) for steep slope (1/15).

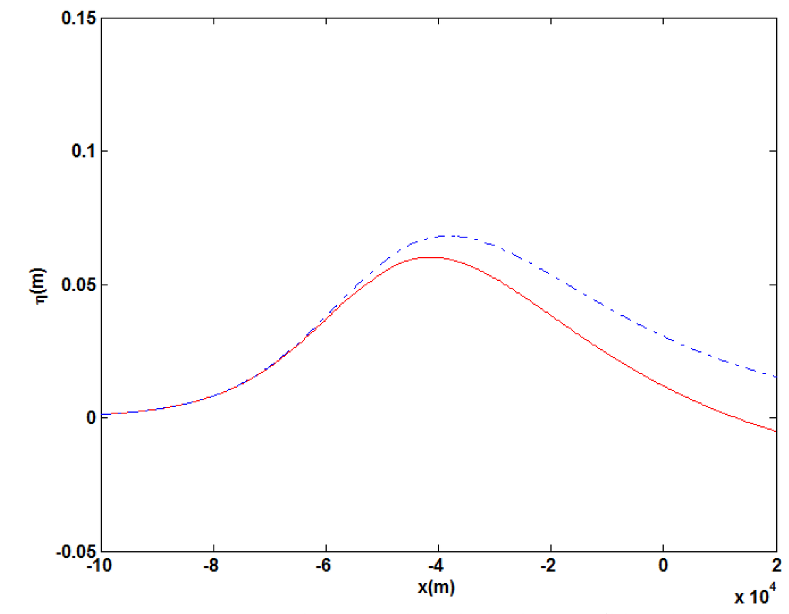

Fig. 8. Comparison between the simulation in the whole domain (blue dashed line) and using EBC (red solid line) for mild slope (1/30).

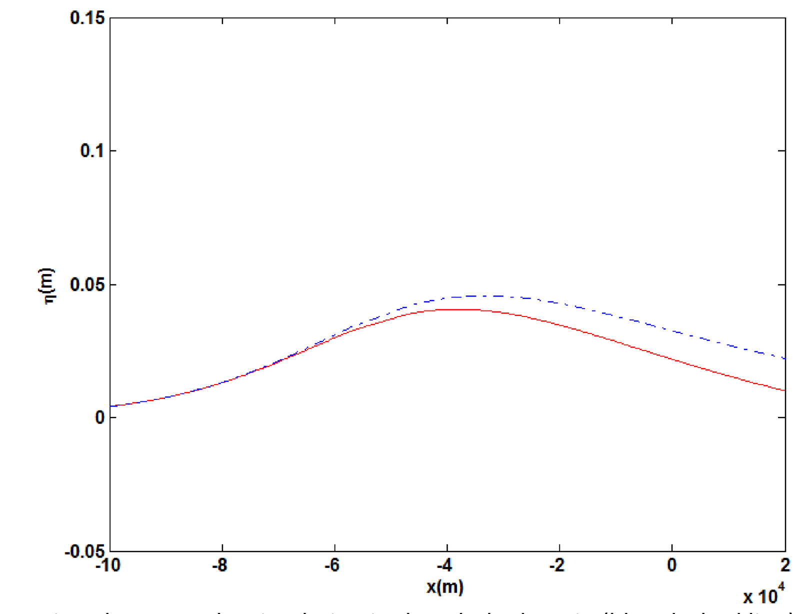

Fig. 9. Comparison between the simulation in the whole domain (blue dashed line) and using EBC (red solid line) for very mild slope (1/45).

Numerical Performance

For the case $w=10km$, the CPU time for solving the PDE when calculated in the whole domain is 86s. The CPU time needed to calculate the reflection wave analytically is 4.56s and the CPU time to solve the PDE in part of the domain, together with influxing the EBC is 69s. This numerical performance shows that the usage of EBC can reduce the computational time.

Acknowledgment

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References

[1] E. van Groesen and J. Molenaar, *Continuum Modeling in the Physical Sciences*, SIAM, 2007.

[2] E. J. Hinch, *Perturbation Methods*, Cambridge University Press, 1991.