HWA 3 - Calculus I – 2018 hand in on Monday May 14, 14.30

1. Find the derivative of the following functions \( h(x) = \sqrt{3 - 2x^2} \)

2. Find the equations of all horizontal and vertical asymptotes of the function

\[
f(x) = \frac{12 - 10x + 2x^2}{x^2 - 3x}
\]

Motivate your answers by showing the appropriate computation/limit.

3. Consider the function \( g(x) = 20x^3 - 3x^5 \) on the domain \([-2, 3]\)
   a. Find the absolute maximum and the absolute minimum of \( g(x) \) (on its domain \([-2, 3]\))
   b. Find the inflection points of \( g \).
   c. Is the function \( g \) even or odd? (Consider the domain = \( \mathbb{R} \) for this question)

4. The function \( f \) is defined as \( f(x) = x^4e^x \).
   a. Find all critical values of \( f \).
   b. Give all local maxima and minima. Motivate each extreme value with the first derivative test.
   c. Give the second derivative of \( f \) and the interval(s), where \( f \) is concave downward.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>2</th>
<th>3a</th>
<th>3b</th>
<th>3c</th>
<th>4a</th>
<th>4b</th>
<th>4c</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.5</td>
<td>2</td>
<td>1.5</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>10</td>
</tr>
</tbody>
</table>

Solutions:
1. \[ h'(x) = \frac{1}{2\sqrt{3} - 2x^2} \cdot -4x = -\frac{2x}{\sqrt{3} - 2x^2} \]

2. The equations of all horizontal and vertical asymptotes of \( f(x) = \frac{12-10x+2x^2}{x^2-3x} \) are:

\[
\frac{2(x^2-5x+6)}{x(x-3)} = \frac{2(x-3)(x-2)}{x(x-3)} \quad \text{Vertical asymptotes } x = a \text{ if } f(a) \text{ has the form } \frac{c}{0}, \text{ where } c \neq 0:
\]

At \( x = 0 \) \( f(x) \) has the form \( \frac{12}{0} \Rightarrow \text{VA: } x = 0 \), but at \( x = 3 \) the form \( 0 \Rightarrow \text{no V.A. at } x = 3. \)

Note that \( f \) has at \( x = 3 \) a removable discontinuity: \[ \lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{2(x-3)(x-2)}{x(x-3)} = \frac{2}{3} \]

Horizontal asymptotes:

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \pm \infty} \frac{12-10x+2x^2}{x^2-3x} = \lim_{x \to \infty} \frac{12}{x^2} \frac{1-\frac{3}{x}}{1-\frac{3}{x}} = 0 = 2
\]

\[
\lim_{x \to -\infty} f(x) = 2 \text{ (similarly) } \Rightarrow \text{HA: } y = 2
\]

3. \( g(x) = 20x^3 - 3x^5 \) has derivative

\[ g'(x) = 60x^2 - 15x^4 = 15x^2(4 - x^2) = 15x^2(x - 2)(x + 2) \]

Using the closed interval method for this continuous function on \([-2, 3]\) we find:

1. \( g'(x) = 0 \Leftrightarrow \text{critical values } x = 0 \text{ or } x = 2 \text{ or } x = -2: \) all within \([-2, 3]\). See table for \( f(x) \).

2. We add the function values \( f(x) \) at the boundaries \( x = -2 \text{ and } x = 3 \) in the following table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>0</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>-64</td>
<td>0</td>
<td>64</td>
<td>-189</td>
</tr>
</tbody>
</table>

3. The largest of these values is the **absolute maximum** \( f(2) = 64 \) and the smallest is the **absolute minimum** \( f(3) = -189 \)

b. \( g''(x) = 120x - 60x^3 = 60x(2 - x^2) = 60x(\sqrt{2} - x)(\sqrt{2} + x) = 0 \)

\Rightarrow x = 0 \text{ or } x = \sqrt{2} \text{ or } x = -\sqrt{2} . \quad \text{Since } x = 0 \text{ and } x = \pm\sqrt{2} \approx \pm1.41 \text{ are all within the interval we have a sign change of } f'' \text{ at all of these values, we have three inflection points: } (0,0), (\sqrt{2},f(\sqrt{2})), (-\sqrt{2},f(-\sqrt{2})).

\( f'(x) = 20x^3 - 3x^5 \Rightarrow g(-x) = -20x^3 + 3x^5 = -g(x): \)

\( g(x) \) is an **odd function**. (the graph can be reflected about the origin.)

4. \( f(x) = x^4e^x \Rightarrow f'(x) = 4x^3e^x + x^4e^x = (x^4 + 4x^3)e^x = x^3(x + 4)e^x = 0 \),

if \( x = 0 \) or \( x = -4 \Rightarrow 0 \text{ and } -4 \text{ are the critical values} \)

b. Since the sign of \( f'(x) \) is changing from positive to negative \( f \) has a **local maximum**

\[ f(-4) = 256e^{-4} \approx 4.69. \]

At \( x = 0 \): the sign of \( f'(x) \) is changing from negative to positive, so \( f \) has a **local minimum** \( f(0) = 0 \)

c. \( f'(x) = (x^4 + 4x^3)e^x \Rightarrow f''(x) = (4x^3 + 12x^2)e^x + (x^4 + 4x^3)e^x = (x^4 + 8x^3 + 12x^2)e^x = x^2(x + 2)(x + 6)e^x \)
\[ f''(x) = 0 \text{ if } \iff x = 0 \text{ or } x = -2 \text{ or } x = -6 \]

\[
\begin{array}{cccccccccc}
\text{sign scheme } f''(x) & + & + & + & 0 & - & - & - & 0 & + & + & 0 & + & + & + \\
\hline
-6 & & & & & & & & & & -2 & & & & & & 0 \\
\end{array}
\]

\[ f''(x) < 0 \text{ if } -6 < x < -2 \text{ the function is concave downward on } (-6, -2) \]

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Some notes on the last exercise (in view of a full investigation and curve sketching):

- In exercise a., to proof that a critical point \( c \) where \( f'(c) \) is a maximum or minimum you have to check whether the sign of \( f'(x) \) is changing at \( x = c \): if \( f' \) is changing from + to -, it is a local maximum, from – to + a local minimum. Similarly in c. to show that a point where \( f''(c) \) is an IP, \( f''(x) \) must change its sign at \( x = c \). e.g. if \( f(x) = x^4 \), then \( f''(x) = 12x^2 = 0 \) if \( x = 0 \), but \((0, 0)\) is not an IP (check the graph!), as \( f''(x) \geq 0 \) near \( x = 0 \)
- If you would use the second derivative test in b. you will find the following, using the second derivative found in c: at \( x = -4 \): \( f''(-4) = 16 \cdot (-2) \cdot (+2)e^{-2} < 0 \Rightarrow f \) is concave downward at \( x = -4 \), so \( f(-4) = 256e^{-4} \approx 4.69 \) is a local maximum. At \( x = 0 \): \( f''(0) = 0 \), but near \( x = 0 \) we have \( f''(x) > 0 \Rightarrow f(0) = 0 \) is a local minimum
- Since \( f''(x) \) has **no sign change** at \( x = 0 \) (“double root”), \((0, 0)\) is not an Inflection Point, IP’s are \((-6, f(-6))\) and \((-2, f(-2))\), where \( f(-6) \approx 3.21 \) and \( f(-2) \approx 2.17 \)
- The behaviour of the function at infinity and negative infinity:
  \[
  \lim_{x \to \infty} x^4e^x = +\infty \text{ (both } x^4 \text{ and } e^x \text{ approach } \infty) \text{ and} \\
  \lim_{x \to -\infty} x^4e^x = \lim_{x \to -\infty} \frac{x^4}{e^{-x}} = 0 \text{ since after substituting } y = -x \text{ we find:} \\
  \lim_{x \to -\infty} \frac{x^4}{e^{-x}} = \lim_{y \to \infty} \frac{y^4}{e^{y}} = 0 \text{ (see exercise 3.14 applying L’Hopital’s rule)}
  \]
- Since \( f(x) \geq 0 \) and the local minimum \( f(0) = 0 \), this is an absolute minimum: There is no absolute maximum since \( \lim_{x \to \infty} x^4e^x = \infty \)