Slides
Probability Theory
Chapter 3

Ch. 3: Conditional Probability and Independence
Overview Basic Probability in Ch. 1+2

Axioms of Kolmogorov: 1. \( P(A) \geq 0 \)

2. \( P(S) = 1 \)

3. \( P(\bigcup_i A_i) = \sum_i P(A_i) \), if the \( A_i \)'s are mutually exclusive.

Complement rule: \( P(\overline{A}) = 1 - P(A) \)

Addition Rule: \( P(A \cup B) = P(A) + P(B) - P(A \cap B) \)

For disjoint events we have: \( P(A \cap B) = 0 \)

Symmetric probability space (Laplace): \( P(A) = \frac{N(A)}{N(S)} \)

number of permutations of \( n \) out of \( N \) is \( \frac{N!}{(N-n)!} \)

number of combinations of \( n \) out of \( N \) is \( \binom{N}{n} = \frac{N!}{n!(N-n)!} \)
Conditional probability

“Probabilities within a part $A$ of the sample space $S$”

Definition: $P(B|A) = \frac{P(A \cap B)}{P(A)}$, if $P(A) > 0$

From this definition the **Product rule** follows:

$P(B \cap A) = P(B|A)P(A)$

or: $P(A \cap B) = P(A|B)P(B)$

“Conditioning w.r.t. $A$ and $B$, respectively”

Product rule for $n$ events:

$P(A_1A_2 \ldots A_n) = P(A_1)P(A_2|A_1) \ldots P(A_n|A_1 \ldots A_{n-1})$

$(S, P(\cdot |A))$ is a probability space (Kolmogorov)
Are foreign and Dutch students equally successful?
A faculty found the following success rates, distinguishing Dutch, European (EER) and non European students:

<table>
<thead>
<tr>
<th>Origin</th>
<th>Non-EER</th>
<th>Dutch</th>
<th>EER</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proportion of students</td>
<td>20%</td>
<td>50%</td>
<td>30%</td>
</tr>
<tr>
<td>Success proportion</td>
<td>80%</td>
<td>60%</td>
<td>90%</td>
</tr>
</tbody>
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Answer the following questions:
1. What is the overall success proportion at the faculty?
2. Determine the probability that a faculty graduate is an EER student.
Solution

- **Define:** \( A = \text{“student succeeds”} \)

  \( S_1 = \text{“The students is non-EER”} \)

  \( S_2 \) and \( S_3 \) similar for Dutch and EER students, resp.

- **Probabilities:** \( P(S_1) = 0.20, \ P(S_2) = 0.50, \ P(S_3) = 0.30, \ P(A|S_1) = 0.80, \ P(A|S_2) = 0.60 \) and \( P(A|S_3) = 0.90 \)

- **Apply the rules** of total probability and Bayes:

  1. \( P(A) = P(A \cap S_1) + P(A \cap S_2) + P(A \cap S_3) \)

     \[ = P(A|S_1)P(S_1) + P(A|S_2)P(S_2) + P(A|S_3)P(S_3) \]

     \[ = 0.80 \times 0.20 + 0.60 \times 0.50 + 0.90 \times 30 = 0.73 \]

  2. \( P(S_3|A) = \frac{P(S_3 \cap A)}{P(A)} = \frac{P(A|S_3)P(S_3)}{P(A)} = \frac{0.9 \times 0.3}{0.73} \approx 0.37 \)
Given:

- \( P(S_i) \) and \( P(A|S_i) \)

Then:

\[
P(A) = P(A \cap S_1) + \cdots + P(A \cap S_k)
\]
\[
= P(A|S_1)P(S_1) + \cdots + P(A|S_k)P(S_k)
\]

Law of total probability

And:

\[
P(S_1|A) = \frac{P(S_1 \cap A)}{P(A)}
\]
\[
= \frac{P(A|S_1)P(S_1)}{P(A|S_1)P(S_1) + \cdots + P(A|S_k)P(S_k)}
\]

Bayes' rule
Independence of events $A$ and $B$:  
“$B$ does not give information about (the probability of) $A$”

\[
P(A|B) = P(A) \iff \frac{P(A \cap B)}{P(B)} = P(A) \iff P(B|A) = P(B)
\]

**Definition:** $A$ and $B$ are independent,  
if $P(A \cap B) = P(A) \cdot P(B)$

$A$, $B$ and $C$ are independent if $P(A \cap B) = P(A) \cdot P(B)$,  
\[
P(A \cap C) = P(A) \cdot P(C)
\]
\[
P(B \cap C) = P(B) \cdot P(C)
\]
and $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$

$A_1, \ldots, A_n$ are independent if the equality holds for any subsequence $A_{i_1}, \ldots, A_{i_k}$ ($k = 2, \ldots, n$)
Independent experiments

Experiments are said to be independent if events related to different experiments are independent. Usually experiments are assumed to be independent.

E.g. Bernoulli experiments or Bernoulli trials:
1. A sequence of independent experiments. (Repetitions of the same experiment)
2. Two possible outcomes: “Success” and “Failure”
3. The “Success”-probability $p$ is always the same.
Application 1 (geometric formula)

$X =$ “the required number of Bernoulli trials until the first success occurs”.

$P(X = k) = (1 - p)^{k-1}p$, where $k = 1, 2, \ldots$

Application 2 (binomial formula)

$X =$ the number of successes in $n$ Bernoulli trials

$P(X = k) = \binom{n}{k} p^k (1 - p)^k, k = 0, 1, \ldots, n$

The numerical variables $X$ are called random variables $\rightarrow$ chapter 4.
Example geometric distribution

Suppose that 10% of the passing cars are Mercedes. 

$X = \text{“the number van the first passing Mercedes”}$

$P(X = k) = 0.9^{k-1} \cdot 0.1$, with $k = 1, 2, 3, \ldots$

We can argue that the expected value is

$E(X) = \frac{1}{0.1} = 10$ and $P(X > 10) = 0.9^{10} \approx 34.9\%$
The binomial distribution – 2 examples

Applicable if we have \( n \) (ind.) Bernoulli trials with success rate \( p \)

**Example:** \( X = \) The number of correct random answers to 10 MC-items:

here \( n = 10 \) and \( p = \frac{1}{4} \)

\[
P(X = x) = \binom{10}{x} \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{10-x}, \ x = 0, \ldots, 10
\]

\( n=10 \) and \( p=0.25 \)

**Example:**

\( X = \) The number of sixes in 30 rolls of a dice.

So \( n = 30 \) and \( p = \frac{1}{6} \)

\[
P(X = x) = \binom{30}{x} \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{30-x}, \ x = 0, \ldots, 30
\]

\( n = 30 \) and \( p = 0.167 \)

What are the expected numbers of correct answers and sixes, resp.?
Summary Chapter 3

• Conditional Probability: \( P(A|B) = \frac{P(A \cap B)}{P(B)} \)

• General product rule: \( P(A \cap B) = P(A|B)P(B) \)

• Product rule in case of independence:
  \[ P(A \cap B) = P(A)P(B) \]

• Law of Total Probability:
  \[ P(A) = P(A|S_1)P(S_1) + \cdots + P(A|S_n)P(S_n) \]

• Bayes` rule:
  \[ P(S_1|A) = \frac{P(S_1 \cap A)}{P(A)} = \frac{P(A|S_1)P(S_1)}{P(A|S_1)P(S_1) + \cdots + P(A|S_k)P(S_k)} \]