Ch. 7: 2 or more continuous variables

- Independence
- The distribution of functions of 2 or more r.v.'s, such as $X + Y$ and $\max(X, Y)$
- The Central Limit Theorem and applications
<table>
<thead>
<tr>
<th>DISCRETE</th>
<th>CONTINUOUS</th>
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<tbody>
<tr>
<td><strong>$S_X$ is numerable</strong></td>
<td><img src="image" alt="Continuous Distribution" /></td>
</tr>
<tr>
<td>probability $P(X = x)$</td>
<td>“probability” $x \cdot f(x) , dx$</td>
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<tr>
<td>$\sum_x P(X = x) = 1$</td>
<td>$\int_{-\infty}^{\infty} f(x) , dx = 1$</td>
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<tr>
<td>$P(a &lt; X &lt; b) =$</td>
<td>$P(a &lt; X &lt; b) = \int_{a}^{b} f(x) , dx$</td>
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<tr>
<td>$\sum_{x \in (a,b)} P(X = x)$</td>
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<tr>
<td>$E(X) = \sum_x x \cdot P(X = x)$</td>
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<tr>
<td>$E(X^2) = \sum_x x^2 \cdot P(X = x)$</td>
<td>$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x) , dx$</td>
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Independence of $X$ and $Y$

**General definition:** $X$ and $Y$ are independent if for all subsets $A$ and $B$ of $\mathbb{R}$ the following equality holds:

$$P(X \in A \text{ and } Y \in B) = P(X \in A) \times P(Y \in B)$$

**Discrete:** choose $A = \{x\}$ and $B = \{y\}$, then

$$P(X = x \text{ and } Y = y) = P(X = x) \times P(Y = y)$$

**Continuous:** simply use the general definition,

**Example:** $X$ and $Y$ are independent and $\text{Exp} \left( \lambda = \frac{1}{2} \right)$

Then: $P(X > 2 \text{ and } Y < 4) = P(X > 2) \times P(Y < 4)$

$$= e^{-\frac{1}{2} \cdot 2} \times \left(1 - e^{-\frac{1}{2} \cdot 4}\right)$$

$$= e^{-1} - e^{-3} \approx 31.8\%$$
Example: maximum of 2 waiting times $X$ and $Y$

- Two customers are served independently: whose service is completed first, $\min(X, Y)$ whose last, $\max(X, Y)$?

- **Model:** waiting times are independent and both $\text{Exp}(\lambda)$.

- $P(\max(X, Y) \leq x) = P(X \leq x \text{ and } Y \leq x)$
  
  \[
  \begin{align*}
  &\overset{\text{ind.}}{=} P(X \leq x) \cdot P(Y \leq x) \\
  &= (1 - e^{-\lambda x})^2 \quad \text{for } x \geq 0
  \end{align*}
  \]

- The distribution function $F_{\max(\, X, Y \,)} (x) = (1 - e^{-\lambda x})^2$

- $\text{En } f_{\max(\, X, Y \,)} (x) = \frac{d}{dx} F_{\max(\, X, Y \,)} (x)$
  
  \[
  = 2(1 - e^{-\lambda x}) \cdot \lambda e^{-\lambda x} , \text{ for } x \geq 0
  \]

- $\min(X, Y)$: \quad $P(\min(X, Y) \geq m) = P(X \geq m \text{ and } Y \geq m)$

  It follows: $f_{\min(\, X, Y \,)} (m) = 2\lambda e^{-2\lambda m}$: $\min(X, Y) \sim \text{Exp}(2\lambda)$
**X + Y: de convolution integral**

\[ P(X + Y = z) = \sum_x P(X = x)P(Y = z - x) \]

\[ f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx \]

**Example:** X and Y are ind. and both \( \text{Exp}(\lambda = 3) \), so:

\[ f_{X+Y}(z) = \int_0^z 3e^{-3x} \cdot 3e^{-3(z-x)} dx \]

\[ = \int_0^z 9e^{-3z} dx = 9e^{-3z} \cdot x \bigg|_{x=0}^{x=z} = 9ze^{-3z} \quad (z \geq 0) \]
The sum of normally distributed variables

Properties of a $N(\mu, \sigma^2)$-distributed $X$ (chapter 6):

- Standardization: $Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$
- $Y = aX + b \sim N(a\mu + b, a^2\sigma^2)$

Consequence of the convolution integral:
If $X$ and $Y$ are independent and
\[ X \sim N(\mu_1, \sigma_1^2) \text{ and } Y \sim N(\mu_2, \sigma_2^2), \]
then
\[ X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2) \]
And
\[ X - Y \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2) \]
Sample distributions

• A random sample from a normal distribution is a sequence of independent and \( N(\mu, \sigma^2) \)-distributed \( X_1, X_2, \ldots, X_n \).

• Then for the sum we have:

\[
\sum_{i=1}^{n} X_i = X_1 + \cdots + X_n \sim N(n\mu, n\sigma^2)
\]

• And for the sample mean:

\[
\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \sim N\left(\mu, \frac{\sigma^2}{n}\right)
\]

• So: \( \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} \)
The distribution of the sample mean

if $X_1, \ldots, X_n$ are ind. all with the same distribution:

- uniform
- exponential
- “odd”

$n = 1$

$n = 2$

$n = 4$

$n = 30$
The Central Limit Theorem (CLT)

for independent and identically distributed $X_1, ..., X_n$ (not normal, but with expectation $\mu$ and variance $\sigma^2$):

$$\lim_{n \to \infty} P\left( \frac{\sum X_i - n\mu}{\sqrt{n\sigma^2}} \leq z \right) = \Phi(z)$$

Note that $\frac{\sum X_i - n\mu}{\sqrt{n\sigma^2}}$ is the result of standardization of $\sum X_i$

Application: for finite, large $n$ (rule of thumb: $n > 25$), we have approximately:

$$\sum_{i=1}^{n} X_i \sim N(n\mu, n\sigma^2) \quad \text{and} \quad \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$
Approximation of \( B(n, p) \) with \( N(\mu, \sigma^2) \)

\( X \sim B(48, \frac{1}{4}) \) approximated by \( Y \sim N(12, 9) \)

The graphs illustrate: \( P(X \leq 15) \approx P\left(Y \leq 15 + \frac{1}{2}\right) \)
Normal approximation of the binomial distribution

Approximate \( X \sim B(n, p) \) by \( Y \sim N(np, np(1-p)) \)

Consequence of the CLT: \( X = \sum_{i=1}^{n} X_i \), where the \( X_i \) are independent alternatives

Condition:
“\( n \) is sufficiently large and \( p \) not too close to 0 or 1”
rule of thumb: \( n > 25 \), \( np > 5 \) and \( n(1 - p) > 5 \).

For this approximation **always apply continuity correction** (c.c.):

\[
P(X \leq k) \overset{c.c.}{=} P(Y \leq k + 0.5) \quad \text{and} \quad P(X < k) \overset{c.c.}{=} P(Y \leq k - 0.5)
\]