Solutions test Calculus I – Functions, Limits, Continuity and Derivatives – April 2018

1. a. \( f(-x) = 2 \cdot (-x) - (-x)^2 = -2x - x^2 \neq f(x) \) or \( f(-x), f \) is odd or even.
   b. x-intercepts: \( f(x) = 0 \iff x(2-x) = 0 \). Hence \( x = 0 \) and \( x = 2 \) are the x-intercepts. y-intercept: \( y = f(0) = 0 \)
   c. Verification: \(-x^2 + 4 = -(x^2 - 4x + 4) + 4 = -x^2 + 4x \neq f(x)\): error in exercise

Transformations of \( y = x^2 \):
1. Shift \( y = x^2 \) two units to the right
2. Reflect \( y = (x-2)^2 \) about the x-axis (or: “stretch” vertically by a factor -1)
3. Shift \( y = -(x-2)^2 \) four units upward.

d. Domain \( D_f = \mathbb{R} \), range \( R_f = (-\infty, 4] \) using \( f(x) = -(x-2)^2 + 4 \) the graph of \( f \) is a parabola, that opens downward with a top \( y = 4 \) at \( x = 2 \) (note that \( y = -(x-2)^2 + 4 \leq 4 \))

e. If \( g(x) = \frac{1}{1-x^2} \), find \( g \circ f(x) = g(2x - x^2) = \frac{1}{1-2x+x^2} \)

\( D_f = \mathbb{R} \) (no restrictions), but the denominator of \( g \circ f \) is \( 1 - 2x + x^2 = (x - 1)^2 \neq 0 \), so \( x \neq 1 \): \( D_{g\circ f} = \mathbb{R} \setminus \{1\} \) (other notations: \{ \( x \in \mathbb{R} \mid x \neq 1 \} \) or \( (-\infty, 1) \cup (1, \infty) \))

f. \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[2(x+h) - x^2] - [2x - x^2]}{h} = \lim_{h \to 0} \frac{2h - 2xh - h^2}{h} \)

\( = \lim_{h \to 0} (2 - 2x - h) = 2 - 2x \)

g. Use the result in f. to find the slope and the equation of the tangent line of the graph of \( f(0) = 0 \): the slope and equation of the tangent line of the graph at \( (0, 0) \) are \( f'(0) = 2 \) and \( y = 2x \), respectively.

2. a. \( \lim_{x \to 5} \frac{x^2 - 25}{x^2 - 10x + 25} = \lim_{x \to 5} \frac{(x-5)(x+5)}{(x-5)^2} = \lim_{x \to 5} \frac{x+5}{x-5} \) d.n.e. (Vertical asymptote at \( x = 5 \))
   b. \( \lim_{x \to 15} \frac{x^2 - 25}{x^2 - 10x + 25} = \lim_{x \to 15} \frac{(x-5)(x+5)}{(x-5)(x-5)} = \lim_{x \to 15} \frac{x+5}{x-5} = -\infty \) (if \( x \downarrow 5, x - 5 \) is close to 0 but negative)
   c. \( \lim_{x \to -5} \frac{x^2 - 25}{x^2 - 10x + 25} = 0 = 0 \) (direct substitution)
   d. \( \lim_{x \to 4} \frac{x^2 + 4x - 32}{2\sqrt{x} - 2} = \lim_{x \to 4} \frac{(x-4)(x+8)(\sqrt{x})}{(2\sqrt{x})} = \lim_{x \to 4} \frac{(x+8)(2\sqrt{x})}{-1} = -48 \)
   e. \( \lim_{x \to \infty} e^{2x} = \lim_{x \to \infty} \frac{1 - e^{-x}}{e^{-2x} + 2} = \frac{1}{2} \) (since e.g. \( \lim_{x \to \infty} e^{-x} = \lim_{x \to \infty} \frac{1}{e^x} = 0 \))
   f. \( \lim_{x \to \infty} \frac{\sin(x)}{x^{\infty}} = 0 \), using the Squeeze Theorem \( -\frac{1}{x} \leq \frac{\sin(x)}{x^2} \leq \frac{1}{x^2} \) and \( \lim_{x \to \infty} \frac{1}{x^2} = 0 \)
   g. \( \lim_{x \to 0} \frac{\cos(x)}{x^2} = +\infty \) (shape \( \frac{1}{x^2} \), where \( x^2 \) is always positive for \( x \) close to 0: VA \( x = 0 \))
   h. \( \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \) (\( \sqrt{x+h} + \sqrt{x} \)) = \lim_{h \to 0} \frac{x+h - x}{h(x+h+x)} = \lim_{h \to 0} \frac{1}{h\sqrt{x+h} + x} = \frac{1}{2\sqrt{x}} \)

3. a. \( \lim_{x \to 10} g(x) = \lim_{x \to 10} 2x = 0 \)
   b. \( \lim_{x \to 0} g(x) = \lim_{x \to 0} x^2 = 0 \) and hence \( \lim_{x \to 0} g(x) = 0 \)
   c. \( \lim_{x \to 1} g(x) = \lim_{x \to 1} (2x - 1) = 1 \)
   d. \( \lim_{x \to -1} g(x) = \lim_{x \to -1} (x^2) = 1 \)
   e. \( \lim_{x \to 0} g(x) = \lim_{x \to 0} x^2 = 0 \)

b. At \( x = 0 \) \( f(x) \) is not defined, so discontinuous, but \( \lim_{x \to 0} g(x) = 0 \) (see a.3.), so it is removable discontinuity.

\( x = 1 \): according to a.4. and 5, \( \lim_{x \to 1} g(x) = 1 = g(1) = 1^2 \); \( f \) is continuous at \( x = 1 \).

c. At \( x = 0 \) \( f(x) \) is not defined (and not continuous), so not differentiable.

d. At \( x = 1 \): \( \lim_{h \to 0} \frac{g(1+h)-g(1)}{h} = \lim_{h \to 0} \frac{(1+h)^2-1}{h} = \lim_{h \to 0} \frac{2h+h^2}{h} = 2 \) and \( \lim_{h \to 0} \frac{g(1+h)-g(1)}{h} = \lim_{h \to 0} \frac{2(h+1)-1}{h} = 2 \). Hence \( g'(1) = \lim_{h \to 0} \frac{g(1+h)-g(1)}{h} = 2 \), so \( g \) is differentiable at \( x = 1 \).