Control of Systems with I/O Delay via Reduction to a One-Block Problem

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Abstract—In this paper the standard (four-block) $H^\infty$ control problem for systems with a single delay in the feedback loop is studied. A simple procedure of the reduction of the problem to an equivalent one-block problem having particularly simple structure is proposed. The one-block problem is then solved by the J-spectral factorization approach, resulting in the so-called dead-time compensator (DTC) form of the controller. The advantages of the proposed procedure are its simplicity, intuitively clear derivation of the DTC form of the controller. The ad-
tantages of the proposed procedure are its simplicity, intuitively clear derivation of the DTC form of the controller, and exten-
tibility to the multiple delay case.

Keywords—Time-delay systems, $H^\infty$ optimization, J-spectral factorization, dead-time compensation.

I. Introduction and problem formulation

Consider the dead-time system in Fig. 1, where $P(s)$ is a finite-dimensional generalized plant with the transfer matrix

$$
P(s) = \begin{bmatrix}
A & B_1 & B_2 \\
C_1 & 0 & D_{12} \\
C_2 & D_{21} & 0 
\end{bmatrix},
$$

where $e^{-sh}$ is the loop delay with the dead-time $h > 0$, and $K_h(s)$ is a proper part of the controller to be designed. The problem to be studied in this paper is formulated as follows:

$\mathcal{OP}_h$: Given the plant $P(s)$ and the dead time $h$, determine whether there exists a proper $K_h(s)$, which internally stabilizes the system in Fig. 1 and guarantees

$$
\|\mathcal{F}(P, e^{-sh}K_h)\|_\infty < \gamma
$$

for a given $\gamma$, and then characterize all such $K_h$ when one exists.

Here $\mathcal{F}(G, U) \equiv G_{11} + G_{12}U(I - G_{22}U)^{-1}G_{21}$ stands for the lower linear fractional transformation of $U$ over $G$, see [1].

$H^\infty$ control of DT systems has been an active research area since mid 80’s. Early frequency response methods, see [2] and the references therein, treated DT systems in the framework of the general infinite-dimensional control theory. This resulted in rather cumbersome solutions, for which implementation and analysis issues appear to be very complicated. This fact motivated more problem-oriented approaches, exploiting the structure of DT systems [3-10], see also the review paper [11] for additional references.

In the late 90’s it was shown [8,12] that suboptimal $H^\infty$ controllers can be presented in the so-called dead-time compensator (DTC) form, i.e., in the form of the feedback interconnection of a finite-dimensional part and an infinite-dimensional “prediction” block reminiscent of the celebrated Smith predictor. The J-spectral factorization approach used in [8,12] produces the DTC form of the controller in an intuitively clear fashion, though the presence of several intermediate steps blurs the final formulae and the relationship with the delay-free problem.

The further simplifications were proposed in [10], where the problem is addressed by the extraction of the dead-time controllers from the known parameterization of the delay-free $H^\infty$ problem. This reduces the four-block problem to a Nehari problem which, in turn, is solved using the results of [13]. The original controller is then recovered in the DTC form as well. The advantage of the result of [10] lies in the transparency and “interpretability” of the resulting controller. Yet the controller recovery is far from being intuitive. This practically prevents the extension of the approach to multiple delay problems.

The purpose of this paper is to amalgamate the approaches of [8] and [10]. As in the latter reference the solution is based on the extraction of the dead-time controllers from the delay-
free parameterization. Yet at this stage the problem is reduced not to a Nehari, but rather to a one-block problem, which turns out to possess some nice properties making it particularly suit-
able for the application of the J-spectral factorization ideas of [8]. This approach allows one to bypass the complicated math needed in the previous approaches and results in probably the simplest solution to date.

Notation. The notation used in this paper is fairly standard. Given a matrix $M$, $M'$ denotes its transpose and $M^{-1}$ stands for $(M')^{-1}$ when the inverse exists. Given a transfer matrix $G(s)$, its conjugate is defined as $G(s)'' = G'(-s)$ and $\|G(s)\|_\infty$ denotes its $H^\infty$ norm (with a slight abuse of notation, it is assumed throughout the paper that $\|G(s)\|_\infty = \infty$ whenever $G(s) \notin H^\infty$). By $\tilde{C}_h(G, U) = (G_{12} + G_{11}U)(G_{22} + G_{21}U)^{-1}$ we denote the chain-scattering (Möbius or homographic) linear fractional transformation.

For a given $G(s) = C(sI - A)^{-1}B$ the h-completion operator $\pi_h\{e^{-sh}G\}$ introduced in [10] is defined as

$$
\pi_h\{e^{-sh}G\} = \tilde{G} - e^{-sh}G = \left[\begin{array}{c}
A \\
C_0
\end{array}\right]
\begin{bmatrix}
A & B \\
C_0 & 0
\end{bmatrix}
\begin{bmatrix}
A & B \\
C_0 & 0
\end{bmatrix}^{-1} - e^{-sh}\left[\begin{array}{c}
A \\
C_0
\end{array}\right].
$$

It can be verified that $\pi_h\{e^{-sh}G\}$ is an entire function of $s$ with the impulse response having support in $[0, h]$ (FIR system).

II. Reduction to one-block problem

A. Solution to the delay-free problem

We start with a brief review of the now classical results on the solvability of the delay-free $H^\infty$ standard problem, i.e., $\mathcal{OP}_0$. To this end, let us impose the following assumptions on the state-

$$
(A1): \quad (C_2, A, B_2) \text{ is stabilizable and detectable;} \quad \text{and} \quad C_2 \quad \text{have full column and row rank, respectively, } \forall \omega \in \mathbb{R};
$$

$$
(A2): \quad \begin{bmatrix} A - j\omega I & B_2 \\
C_1 & D_{12}
\end{bmatrix} \text{ and } \begin{bmatrix} A - j\omega I & B_1 \\
C_2 & D_{21}
\end{bmatrix}
$$

have full column rank and row rank, respectively; and

$$
(A3): \quad D_{12}C_1 + D_{12} = \left[\begin{array}{c}
0 \\
I
\end{array}\right] \quad \text{and} \quad D_{21}B_1 + D_{21} = \left[\begin{array}{c}
B_1 \\
0
\end{array}\right].
$$
Introduce also the following $H^\infty$ algebraic Riccati equations:

$$XA + A'X + C_1'C_1 - XB_1B_2'X + \gamma^2 XB_1'X = 0 \quad (1)$$

and

$$AY + YA' + B_1B_1' - YC_2'C_2Y + \gamma^2 YC_1'C_1Y = 0. \quad (2)$$

The solutions to Riccati equations (1) and (2) are said to be stabilizing if the matrices $A_F \triangleq A + \gamma^2 B_1'B_2X - B_2'B_2X$ and $A_L \triangleq A + \gamma^2 YC_1'C_1 - YC_2'C_2$, respectively, are Hurwitz. Then [1] $\mathcal{OP}_0$ is solvable iff 
(a) there exists a stabilizing solution $X = X' \geq 0$ to ARE (1);
(b) there exists a stabilizing solution $Y = Y' \geq 0$ to ARE (2);
(c) $\rho(XY) < \gamma$.

Furthermore, if these conditions hold, then the transfer matrix $G_0(s)$ can be formulated as

$$G_0(s) = \begin{bmatrix} A_F & ZB_2 & ZYC_2' \\ -B_2X & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad (3)$$

where $Z \triangleq (I - \gamma^2 YX)^{-1}$, is well defined and the set of all admissible controllers is parametrized as $K_0 = C_1(G_0, Q_0)$, where $Q_0$ must satisfy $\|Q_0\|_\infty < \gamma$ but otherwise arbitrary.

**Remark 1:** Note that by construction the matrix $A_F$ in (3) is Hurwitz. Moreover, the “$A$” matrix of $G_0^{-1}$,

$$A_F + ZB_2B_2'X - ZYC_2'C_2 = ZA_LZ^{-1}, \quad (4)$$

so it is Hurwitz as well. Hence, $G_0$ given by (3) is bistable.

**B. From standard problem to one-block problem**

The parameterization of all admissible controllers given above can be visualized as shown in Fig. 2(a). The key property of the mapping $Q_0 \mapsto K_0$ for $G_0$ given by (3) is that it is an isomorphism, so that $K_0 \triangleq C_1(G_0, Q_0) \iff Q_0 = C_0(G_0^{-1}, K_0)$, see Fig. 2(b). It then follows that provided conditions (a)-(c) above hold, a controller $K_0$ solves $\mathcal{OP}_0$ iff $\|C_1(G_0^{-1}, K_0)\|_\infty < \gamma$.

On the other hand, the delay can be thought of as just an additional restriction imposed upon the controller $K_0$. This means that (for any $h > 0$) $\mathcal{OP}_0$ is solvable only if so is $\mathcal{OP}_0$. Therefore, combining the parameterization of all solutions to $\mathcal{OP}_0$ with the transformation in Fig. 2 the following result can be formulated:

**Lemma 1:** $\mathcal{OP}_0$ is solvable iff so is its delay-free counterpart $\mathcal{OP}_h$ and, in addition, $\|C_1(G_0^{-1}, e^{-ah}K_0)\|_\infty < \gamma$.

Lemma 1 actually implies that $\mathcal{OP}_0$ can be converted to the following equivalent problem:

$\mathcal{OP}_0$: Given the bistable system $G_0(s)$ with the state-space realization (3) and the dead time $h$, determine whether there exists a proper $K_h(s)$, which guarantees

$$\|C_1(G_0^{-1}, e^{-sh}K_h)\|_\infty < \gamma$$

for a given $\gamma$, and then characterize all such $K_h$ when one exists.

Note that $G_0^{-1}$ partitioned according to the signal partition in Fig. 2(b) has “square” (1, 1) and (2, 2) blocks. Hence $\mathcal{OP}_0$ falls into the class of the so-called one-block problems, the solution to which is simpler than that to $\mathcal{OP}_0$. In other words, Lemma 1 reduced the general (four-block) problem $\mathcal{OP}_0$ to a simpler one-block problem $\mathcal{OP}_h$. Moreover, only IO (rather than internal) stability is required for the system in Fig. 2(b), which may simplify the analysis.

**Remark 2:** It is worth stressing that the reasoning above applies to any constrained version of the standard problem. Thus, any four-block problem with some constraints imposed on the controller (i.e., multiple delay problems) can be reduced to a one-block problem in a simple and intuitive way.

**III. Solution to the one-block problem**

**A. The main results**

We start with the formulation of the solution to $\mathcal{OP}_h$. Toward this end the following symplectic matrix function is required:

$$\Sigma(t) = \begin{bmatrix} \Sigma_{11}(t) & \Sigma_{12}(t) \\ \Sigma_{21}(t) & \Sigma_{22}(t) \end{bmatrix} = \exp\left(\begin{bmatrix} A_F + ZB_2B_2'X & ZYC_2'C_2Y' \\ -XZB_2B_2'X & -A_F - XZB_2B_2'Z' \end{bmatrix} t \right). \quad (5)$$

For the sake of simplicity, hereafter we use $\Sigma$ to mean $\Sigma(h)$. Introduce also the quantity

$$\gamma_h \triangleq \left\| \begin{bmatrix} A_F + ZB_2B_2'X & ZYC_2'C_2 \\ -XZB_2B_2'X & 0 \end{bmatrix} \right\|_{L^2[0,h]} \quad (6)$$

This is the $L^2[0,h]$-induced norm of an LTI system—a notion extensively studied in the delay and sampled-data literature, see [2, 14] and the references therein. It is well known [14] that $\gamma > \gamma_h$ iff $\Sigma_{22}(t)$ is nonsingular for all $t \in [0,h]$.

We are now in the position to formulate our main result:

**Theorem 1:** $\mathcal{OP}_h$ is solvable iff $\gamma > \gamma_h$. In that case all solutions $K_h$ to the $\mathcal{OP}_h$ are given by

$$K_h = C_1\left(\left[\begin{array}{c} \gamma \end{array}\right]G_h, Q_h\right) \quad (7)$$

(see Fig. 3), where

$$\Delta = \pi_h \left\{ \begin{array}{c} A_F + ZB_2B_2'X & ZYC_2'C_2Y' \\ -XZB_2B_2'X & -A_F - XZB_2B_2'Z' \end{array} \right\}, \quad (8)$$

$$G_h = \left[\begin{array}{c} A_F \\ -B_2X \\ C_2(\Sigma_{22} - \frac{1}{\gamma} YZ\Sigma_{21}) \end{array}\right] \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \left[\begin{array}{c} \Sigma_{12}^{-1} \Sigma_{22} \end{array}\right], \quad (9)$$

and $Q_h$ must satisfy $\|Q_h\|_\infty < \gamma$ but otherwise arbitrary.

Having this result, the solution to $\mathcal{OP}_h$ can now be formulated as follows:
Corollary 1: If $O_h$ is solvable if is so is $O_{h0}$ and also $\gamma > \gamma_h$. In that case all solutions $K_h$ to the $O_{h0}$ are given by (7).

Remark 3: The formulae for $\Sigma(t)$ and $\Delta$ could be further cleaned up as shown in [10]. The reader could also find there the more conventional LFT form of parametrization (7).

The rest of this section is devoted to the proof of Theorem 1. In $\S$II-B we outline the main ideas of the proof, then, in $\S$III-C, we introduce some technical machinery to be used in the sequel, in $\S$III-D we derive the necessary conditions for solvability of $O_{h0}$, $\S$III-E is devoted to the construction of $\Delta$ and $G_h$, and, finally, in $\S$III-F we prove the validity of the formulae.

B. Proof outline

In the proof of Theorem 1 we use the $J$-spectral factorization approach. Let $J_\gamma = \begin{bmatrix} I & 0 \end{bmatrix}$. We are looking for a bistable $W_h$ so that

$$W_h = G_0 J_\gamma G_0 = W_{h0} J_\gamma W_{h0},$$

and $G_0 W_{h0}^{-1}$ is $J_\gamma$-lossless, see [15, 16] for the definitions. If $G_0$ were finite dimensional, then the existence of such a $W_h$ would be necessary and sufficient for the solvability of $O_{h0}$ and the set of all solutions would be parameterized by $K_h = C_0 \begin{bmatrix} W_{h0}^{-1} & Q_h \end{bmatrix}$ with $\|Q_h\|_\infty < \gamma$. Yet $G_0$ is infinite dimensional. This complicates both the construction of $W_h$ and the proof that the factorization above does yield the solution to $O_{h0}$. To circumvent this obstacle the approach of [8] is used. The idea is to exploit the special structure of $P_{h0}$ and use it to remove the infinite dimensional part from the factorization. To this end note that the infinite dimensional part of $P_{h0}$ only enters the off-diagonal blocks,

$$P_{h0} = \begin{bmatrix} P_{11} & e^{-s\beta} P_{12} \\ e^{s\beta} P_{21} & P_{22} \end{bmatrix}.$$  

Here $P_{1j}$ are the subblocks of the (finite-dimensional) transfer matrix $P = \begin{bmatrix} G_0^{-1} & J_\gamma G_0 \end{bmatrix}$. Also note that the $J_\gamma$-spectral factorization of $P_{h0}$ can be reduced to that of

$$P_{h0} = \begin{bmatrix} P_{11} & e^{-s\beta} P_{12} \\ e^{s\beta} P_{21} & P_{22} \end{bmatrix},$$

provided $\Delta \in H^\infty$. Indeed, one can see that $W_h$ is a bistable $J_\gamma$-spectral factor of $P_{h0}$ iff

$$W_h = W_h \begin{bmatrix} I & 0 \end{bmatrix},$$

is a bistable $J_\gamma$-spectral factor of $P_{h0}$. The idea then is to choose $\Delta$ so as to make $P_{h0}$ finite dimensional. It is easy to verify that

$$P_{h0} = \begin{bmatrix} P_{11} - e^{-s\beta} P_{12} P_{12} R^{-1} P_{22} R^{-1} P_{22} \\ R^{-1} P_{22} R^{-1} P_{22} R \end{bmatrix}$$

for $\Delta = \Delta + e^{-s\beta} P_{12} P_{12}$. This $R$ is finite dimensional if we choose $\Delta$ to be the stable FIR system

$$\Delta = \pi_h \{ e^{-s\beta} P_{12} P_{12} \},$$

(incidentally, $P_{22}$ is invertible because of the structure of $G_0$). In $\S$III-E we show that this choice yields the $\Delta$ of Theorem 1 and that $G_0$ defined in Theorem 1 equals $G_0 = W_{h0}^{-1}$ where $W_{h0}$ is a finite dimensional $J_\gamma$-spectral factor of $P_{h0}$.

Typically it is the existence of such $G_h = W_{h0}^{-1}$ that forms the bottleneck of the proof. Here, however, we bypass this difficulty by first showing that $\gamma$ must exceed $\gamma_h$ if $O_{h0}$ is to have a solution, see $\S$III-D. Therefore $\gamma > \gamma_h$ and this guarantees invertibility of $\Sigma_{22}$ and, hence, existence of $G_h$ (see Theorem 1). With $G_h$ known to exist the rest of the proof follows fairly standard arguments. Continuity is used to show that $G_h W_{h0}^{-1}$ is $J_\gamma$-lossless. Finally, as in the finite dimensional case, all solutions $K_h$ are shown to have the form $K_h = C_0 \begin{bmatrix} W_{h0}^{-1} & Q_h \end{bmatrix} = C_0 \begin{bmatrix} I & W_{h0}^{-1} \end{bmatrix}$ with $\|Q_h\|_\infty < \gamma$.

C. Preliminary: $S$-transformations

Throughout this section we will extensively use the “Schur complementation” transformations $S_u(O)$ and $S_r(O)$, which are defined for a $2 \times 2$ block operator $O$ as follows:

$$S_u(O) = \begin{bmatrix} O_{11} - O_{12} O_{21}^{-1} O_{12} & -O_{11} O_{12} \\ O_{21} O_{12}^{-1} O_{21} & O_{12} \end{bmatrix},$$

$$S_r(O) = \begin{bmatrix} O_{11} - O_{12} O_{21}^{-1} O_{21} O_{12} & -O_{12} O_{21}^{-1} O_{21} \\ -O_{12} O_{21}^{-1} O_{21} & O_{21} O_{12}^{-1} \end{bmatrix}.$$  

In the sequel we call these transformations the upper and lower $S$-transformation, respectively. It is clear that the upper (lower) $S$-transformation is well-defined if the upper left (lower right) subblock of $O$ is nonsingular. $S$-transformations can be thought of as the “swapping” of parts of the inputs and outputs, namely

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = O \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \iff \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = S_u(O) \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \iff \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = S_l(O) \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}$$

(provided the mappings are well-defined). The relations above prompt an elegant way to perform $S$-transformations for systems given by their state-space realizations. Indeed, if

$$\Phi(s) = \begin{bmatrix} A_0 & B_{01} & B_{02} \\ C_{01} & I & 0 \\ C_{02} & 0 & I \end{bmatrix},$$

then the straightforward flow-tracing yields

$$S_u(\Phi(s)) = \begin{bmatrix} A_0 - B_{01} C_{01} & B_{01} & B_{02} \\ C_{01} & I & 0 \\ C_{02} & 0 & I \end{bmatrix},$$

$$S_l(\Phi(s)) = \begin{bmatrix} A_0 - \kappa B_{02} C_{02} & B_{01} & \kappa B_{02} \\ C_{01} & I & 0 \\ -\kappa C_{02} & 0 & I \end{bmatrix}.$$  

The signal swapping interpretation implies also the following relations when corresponding transformations exist:

$$S_u(S_u(O)) = S_r(S_l(O)) = O,$n
$$S_u(S_l(O)) = S_r(S_u(O)) = O^{-1}.$$  

See also [16, Ch. 4], where similar transformations were introduced.

Another advantage of looking at the $S$-transformation of $O$ instead of at $O$ itself is that

$$S_r(\begin{bmatrix} \delta & 0 \\ 0 & \Delta \end{bmatrix} O \begin{bmatrix} \delta & 0 \\ 0 & \Delta \end{bmatrix}) = S_r(O) \begin{bmatrix} \delta & 0 \\ 0 & \Delta \end{bmatrix}.$$  

This relation will be used in Subsection III-E.

D. Necessary solvability conditions

We start with finding the necessary condition for the solvability of $O_{h0}$. To this end, note that given any proper $K_h$, the responses of $C_0(G_0^{-1}, e^{-s\beta} K_h)$ and $C_0(G_0^{-1} 0)$ to any input coincide in the interval $[0, h]$ (the former system is actually

...
In this subsection the assumption above is required to ensure 
\[
   \gamma > \|C_r(G_0^{-1}, 0)\|_{L^2[0, h]}.
\]
To find the state-space realization of \( C_r(G_0^{-1}, 0) \), note that it is equal to the (1, 2)-subblock of \( S_1(G_0^{-1}) \). Yet, according to (9), \( S_r(G_0^{-1}) = S_4(G_0) \). This, together with (8a), yields that
\[
   C_r(G_0^{-1}, 0) = \begin{bmatrix}
   A_F + ZB_1B_2^TZY C_2^2 \\
   B_2Z \\
   C_2Y泽
   \end{bmatrix}
\]
We thus proved the following result:

**Lemma 2.** \( \mathcal{O}_B \) is solvable only if \( \gamma > \gamma_h \), where \( \gamma_h \) is the \( L^2[0, h] \)-induced norm of \( C_r(G_0^{-1}, 0) \) given by (6).

**E. Factorization of \( G_0^*J_1, G_0 \)**

By Lemma 2 we can safely assume throughout that \( \gamma > \gamma_h \).
In this subsection the assumption above is required to ensure that \( \Sigma_{22} \) is invertible.

Consider \( \Pi = (G_0^{-1})^{-1} J_1, G_0^{-1} \), which has the following state-space realization (recall (4)):
\[
   \Pi = \frac{Z A_L Z^{-1}}{Z B_2 Z \gamma C_2} + \begin{bmatrix}
   Z B_2 Z \gamma C_2 \\
   B_2Z \\
   C_2Y泽
   \end{bmatrix}
\]
Note that by construction
\[
   [B_{11} B_{12}] = [\Pi_{11} \Pi_{12}].
\]
(11)
where \( \Pi = [0 \; I] \).

Now for the construction of \( \Delta = \pi_\theta \{ e^{-s\theta} \Pi_{22}^{-1} \Pi_1 \} \) we use the fact that \( -\Pi_{22}^{-1} \Pi_2 \) is the lower left block of \( S_4(\Pi) \). Therefore consider (using (8b))
\[
   S_4(\Pi) = \frac{A_{11} + \frac{1}{\gamma} B_{12} C_{12}}{\pi_\theta} \frac{B_{11} - \frac{1}{\gamma} B_{12} C_{12}}{0 - \frac{1}{\gamma} I}
\]
(4) one can see that \( \Sigma(t) = e^{(A_{11} + \gamma^{-2} B_{12} C_{12})t} \). Then
\[
   \Delta = \pi_\theta \left\{ e^{-s\theta} \left[ \frac{A_{11} + \gamma^{-2} B_{12} C_{12}}{0 - \frac{1}{\gamma} C_{12}} \right] \right\}
\]
and this coincides with the realization in Theorem 1. Moreover,
\[
   R \equiv \Delta + e^{-s\theta} \Pi_{22}^{-1} \Pi_1 = \begin{bmatrix}
   A_{11} + \frac{1}{\gamma} B_{12} C_{12} \\
   \frac{C_{11} \Sigma - B_{11} - \frac{1}{\gamma} B_{12} C_{12}}{0 - \frac{1}{\gamma} I}
   \end{bmatrix}
\]
Next, use (10) and then combine the various realizations:
\[
   S_4(\Pi_2) = S_4 \left( \left[ \begin{array}{c}
   0 \\
   0 \\
   \frac{A_{11} + \frac{1}{\gamma} B_{12} C_{12}}{0 - \frac{1}{\gamma} C_{12}} \\
   \frac{B_{11} - \frac{1}{\gamma} B_{12} C_{12}}{0 - \frac{1}{\gamma} I}
   \end{array} \right] \right)
\]
by (10)
\[
   = \frac{A_{11} + \frac{1}{\gamma} B_{12} C_{12}}{\frac{C_{11} \Sigma - B_{11} - \frac{1}{\gamma} B_{12} C_{12}}{0 - \frac{1}{\gamma} I}}
\]
This, together with (9a) and (8b), yields
\[
   \Pi_\beta = \begin{bmatrix}
   A_{11} \\
   C_{11} \Sigma \\
   \frac{C_{11} B_{11}}{0 - \frac{1}{\gamma} I}
   \end{bmatrix}
\]
and the “\( B \)” and “\( C \)” matrices of \( \Pi_\beta \) satisfy
\[
   \begin{bmatrix}
   \Sigma^{-1} B_{11} \\
   B_{12}
   \end{bmatrix} = \begin{bmatrix}
   0 \\
   1
   \end{bmatrix}
\]
which follows from (11) and the fact that \( \Sigma \) is symmetric and thus \( \Sigma J_2 = \Sigma \).

To \( J \)-factorize \( \Pi_\beta \), let \( M = M' \) be any matrix satisfying the following Riccati equation
\[
   \begin{bmatrix}
   -M & I & \Sigma^{-1} (A_{11} - B_{11} C_{11} + \frac{1}{\gamma} B_{12} C_{12})
   \\
   \Sigma^{-1} B_{11} & B_{12} & \Sigma
   \end{bmatrix} = \begin{bmatrix}
   I & 0 \\
   0 & I
   \end{bmatrix}
\]
(we construct one such \( M \) below). Then, taking into account (12), one can verify that stable
\[
   \Pi_\beta = \begin{bmatrix}
   Z A_L Z^{-1} \\
   \frac{C_{11} \Sigma}{J_2} \\
   \frac{C_{11} \Sigma}{J_2}
   \end{bmatrix}
\]
does satisfy \( \Pi_\beta = W_\gamma J_1, W_\beta \). Moreover, the “\( A \)” matrix of \( W_\beta \), say \( A_\beta \), satisfies
\[
   A_\beta = \frac{Z A_L Z^{-1} - (I \; 0) \Sigma^{-1} B_{11} C_{11} - \frac{1}{\gamma} B_{12} C_{12}}{I \; M}
\]
(12)
and \( \Sigma \Sigma^{-1} \Sigma \) is bistable. Straightforward algebra yields then that \( W_\beta^{-1} = G_h \), where \( G_h \) as in Theorem 1.

**F. Necessity \& sufficiency**

By construction we have that \( \|G_r(G_0^{-1}, e^{-s\theta} K_h)\|_{L^\infty} < \gamma \) iff
\[
   \|Q_h\|_{L^\infty} < \gamma \text{ for } Q_h \text{ defined as } Q_h = C_r \left( G_h^{-1} \left[ \begin{array}{c}
   -\gamma \iota \\
   0
   \end{array} \right], K_h \right).
\]

Now this \( Q_h \) is proper if \( K_h \) is proper, yet the set of proper operators in \( L^\infty \) is in fact \( H^\infty \), [17] (see also [18, A6.26.c, A6.27]).
So \( K_h \) solves \( \mathcal{O}_B \) then necessarily \( \|Q_h\|_{L^\infty} < \gamma \). This condition on \( Q_h \) is also sufficient as we shall now see. The thing to note is that
\[
   \Theta_\theta(h) = \lim_{h^{-2} \Pi_{22}^{-1} \Pi_1} \left( \begin{array}{c}
   \delta \\
   \delta
   \end{array} \right)
\]
is not only stable and \( J \)-unitary (i.e., \( \Theta_\theta(h) J_1, \Theta_\theta(h) = J_1 \)) but in fact \( J \)-lossless (meaning that in addition \( \Theta_\theta(h) \) is bistable).
Indeed, from \( \Theta(h) J_1, \Theta(h) \) = \( J_1 \), it follows that \( \Theta_\theta(h) \circ \Theta_\theta(h) \geq I \), and as our \( \Theta(t) \) (which by Lemma 2 exists for all \( \gamma > \gamma_h \)) is stable and continuous as a function of \( t \in [0, h] \), and \( \Theta_\theta(0) = I \) it follows that \( \Theta_\theta(h) \) is bistable. It is well known that for \( J \)-lossless \( \Theta(h) \) we have that \( Q = C_r \left( \Theta(h), Q_h \right) \) is stable for any \( \|Q_h\|_{L^\infty} < \gamma \), see, e.g., [8, Thm. 6.2]. Also, \( K_h = C_r \left( \left[ \begin{array}{c}
   \delta \\
   \delta
   \end{array} \right], G_h, Q_h \right) \) is proper for any stable \( Q_h \).
IV. Concluding remarks

In this note two “competing” approaches to $H^\infty$ control for systems with a single delay have been put together and the result is probably the simplest solution to date to the problem. Instrumental is the idea to reduce the problem to a one-block problem with a simple structure. In fact in the mean time this idea has been put to use to solve the case where there are multiple delays. This will be reported elsewhere.

References