Convergence Tests for the Hamiltonian Particle-Mesh Method

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Outline

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Why looking at convergence aspects of HPM?

- Many numerical test cases show that method is convergent, see e.g. Frank et al. (2002).
- However, not really documented:
  - Order of accuracy?
  - Restrictions in choice HPM parameters to assure convergence?

Key aspects of Hamiltonian Particle-Mesh Method (HPM)

1. Based on a parcel formulation (Bokhove & Oliver, 2006).
2. In the parcel formulation the fluid is regarded as continuum of fluid parcels in a Eulerian prescribed potential field.
3. A grid allows for computationally efficient smoothing.

Parcel Formulation for SWE

Label each parcel in continuum by its initial position $\boldsymbol{A}$.
Position $\boldsymbol{X}(t)$ and velocity $\boldsymbol{V}(t)$ for parcel $\boldsymbol{A}$ are found from

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \boldsymbol{\nabla}_{\boldsymbol{V}} H = V \,, \tag{1a}$$

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = -f\, \boldsymbol{\nabla}^{\perp}_{\boldsymbol{V}} H - \boldsymbol{\nabla}_{\boldsymbol{X}} H = -f\, \boldsymbol{V}^{\perp} - g \boldsymbol{\nabla}_{\boldsymbol{X}} h \,, \tag{1b}$$

$(\boldsymbol{\nabla}_{\boldsymbol{X}}, \boldsymbol{\nabla}_{\boldsymbol{V}}) \equiv (\frac{\partial}{\partial \boldsymbol{X}}, \frac{\partial}{\partial \boldsymbol{V}}),$

with Parcel Hamiltonian

$$H(\boldsymbol{X}, \boldsymbol{V}, t) = \frac{1}{2} \boldsymbol{V}^2 + g\, h(x, y, t)\big|_{(x,y)=\boldsymbol{X}} \tag{2}$$

Continuity relation closes the continuum description:

$$h(x, y, t) = \int h(\boldsymbol{A}, 0)\, \delta((x, y)) - \boldsymbol{\chi}^t(\boldsymbol{A}))\, \mathrm{d}\boldsymbol{A}. \tag{3}$$

- For more information on parcel formulation: Marcel Oliver and Onno Bokhove.

The Hamiltonian Particle-Mesh Method (HPM)

(Frank et al., 2002)

- Based on the parcel formulation, so that $h(\boldsymbol{X}, t)$ is computed from density distribution on the grid.

The algorithm:

i) Fluid decomposed in N particles.

ii) **Lagrangian step**:
Apply symplectic time integrator to move the particles.

iii) **Eulerian steps**:

a. Fixed grid (x_i, y_j).

b. Redistribution of particles induces new density field.
Density on grid found from discretization of (3).

c. Interpolate grid values $h_{ij}(t)$ back to $\bar{h}(x, y, t)$ using appropriate interpolation function.

iv) **Spectral step**:

Apply smoothing operator to $h_{ij}(t)$ for numerical stability.

So, we have following **HPM parameters** to play with:

- NCx, NCy: number of cells in x and y direction.
- Total number of particles (N), specified as initial number $\Gamma_{x,y}$ of particles per cell in x, y direction:
  $N = (\Gamma_x \, NCx) \, (\Gamma_y \, NCy)$.
- Smoothing length μ.

Test: Burger's profile

Assume

- 1-dimensional flow, $f = 0$.
- $u(x,t) + 2\sqrt{g\,h(x,t)} = K$, with K constant.

Then the SWE reduces to Burgers' equation

$$\frac{\partial q}{\partial t} + q\frac{\partial q}{\partial x} = 0$$

for $q(x,t) = K - 3\sqrt{g\,h(x,t)}$.

→ Analytical solution implicitly defined by $q(x,t) = q_0(x - q(x,t)\,t)$.
→ Wave breaking after certain time.

Spatial convergence rate?

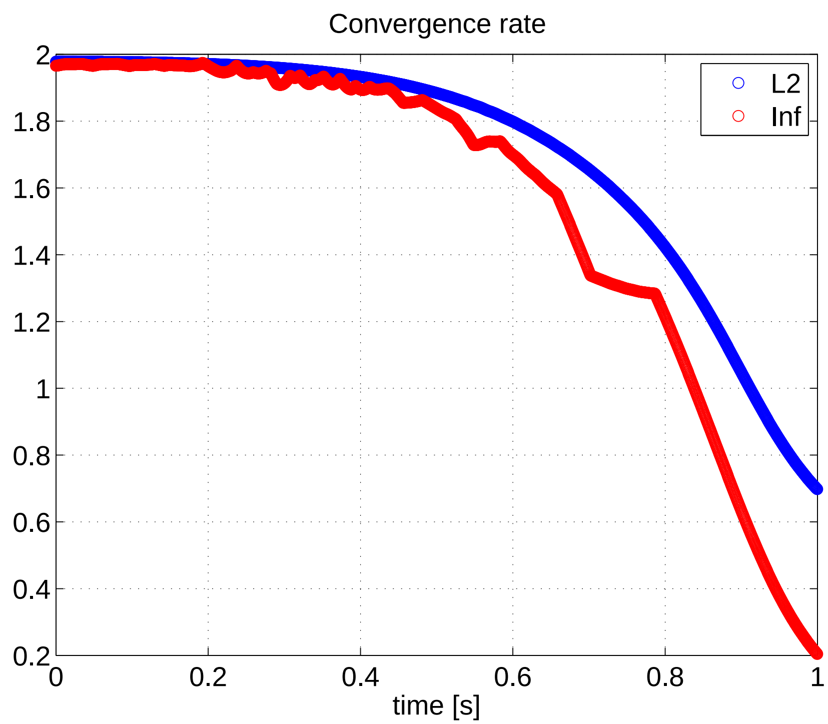
Input

- Smooth initial profile,
- second order symplectic time integrator,
- qubic spline interpolation functions.

$\Rightarrow$ spatial convergence rate can be **2 at the most**.

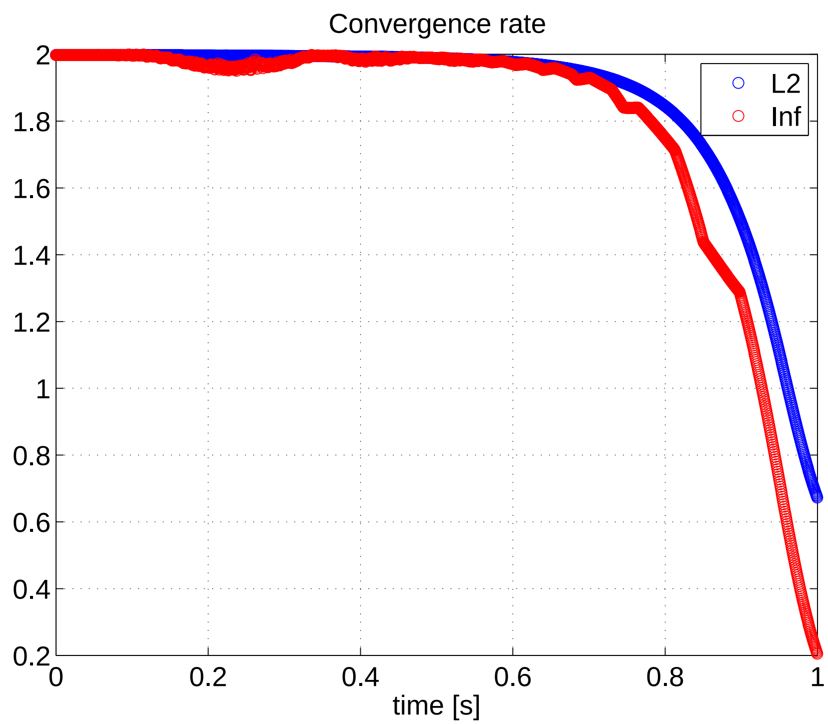
Test 1:

- Grid refinement $NCx = 50 \rightarrow 100$.
- $\Gamma_x = 2$,
- $\mu = \Delta x$.



Test 2:

- Grid refinement $NCx = 200 \rightarrow 400$.
- $\Gamma_x = 4$,
- $\mu = \Delta x$.



Test: Iacono solution

Following Iacono (2005), we can construct steady state analytic nonlinear solutions to the SWE with bottom topography.

Tested solution:

$\vec{v}(x,y) = \alpha\,(\beta\,\sin y, -\sin x),$

$h(x,y) = \frac{f}{g}\,\alpha\,(\cos x + \beta\,\cos y) + h_0,$

$b(x,y) = -\frac{\alpha^2\,\beta}{g}\,\cos x\,\cos y$

on domain $[0, 2\pi]^2$.

$\Rightarrow$ Vorticity profile given by $\zeta(x,y) = \alpha\,(\cos x + \beta \cos y)$.

Input

- $(\alpha, \beta) = (0.1, 1)$,
- $h_0 = 0.3$ assuring $h > 0$.

Of course, our solution is not steady state in Lagrangian frame. So interesting question is:

How does HPM behave for this solution?

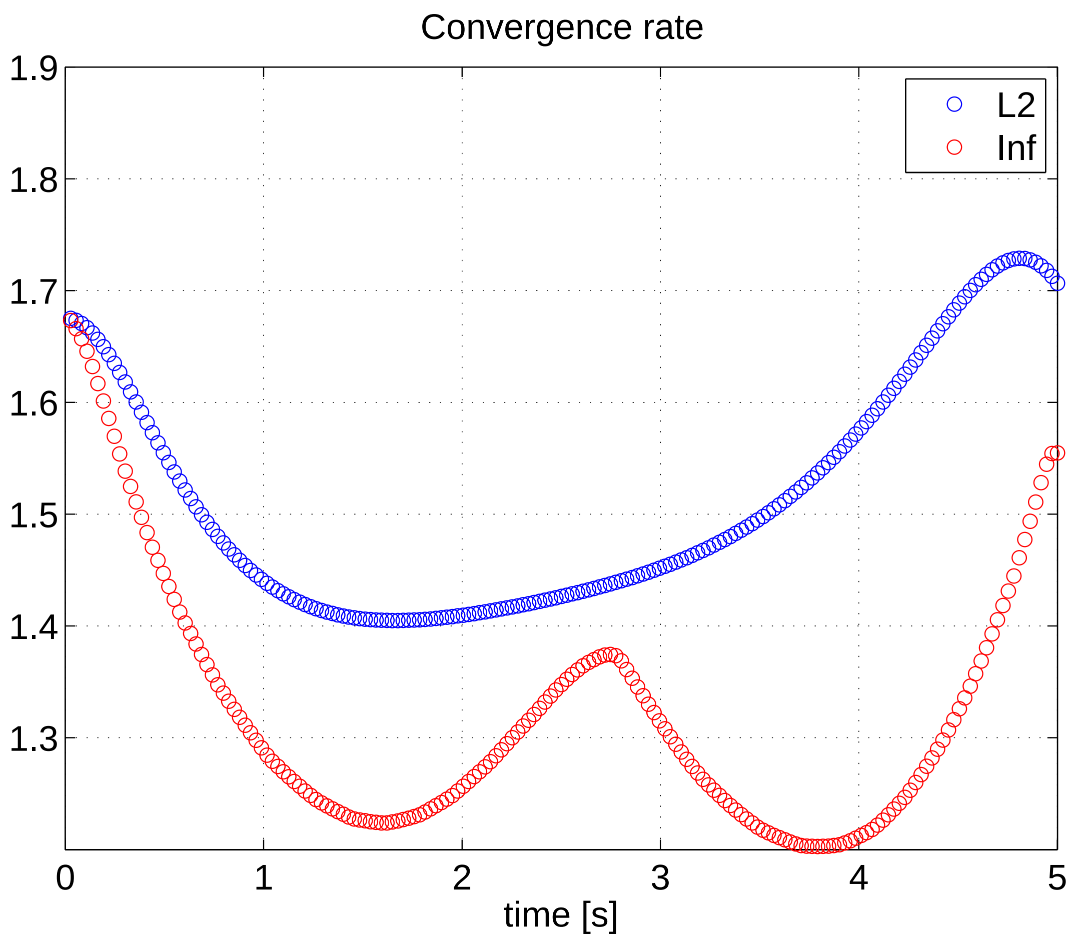
$\Rightarrow$ Movie

- $NCx = NCy \sim 260$,
- $\Gamma_x = \Gamma_y \sim 4$,
- $\mu \sim 6\,\Delta x$.

Spatial convergence rate?

Test:

- Grid refinement $NCx = 40 \rightarrow 80 \rightarrow 160$,
- $NCy = NCx$,
- $\Gamma_x = \Gamma_y = 4$,
- $\mu = 4\,\Delta x$.


Convergence rate
1.9
1.8
1.7
1.6
1.5
1.4
1.3
0
1
2
3
4
5
time [s]
L2
Inf