$$\textbf{Fluid} = \sum \textbf{particles}$$

A Hamiltonian Numerical Scheme for the Hydrostatic Euler Equations

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Motivation

- Why Hamiltonian scheme?
- Indeed: in real atmosphere forcing and friction
- But,

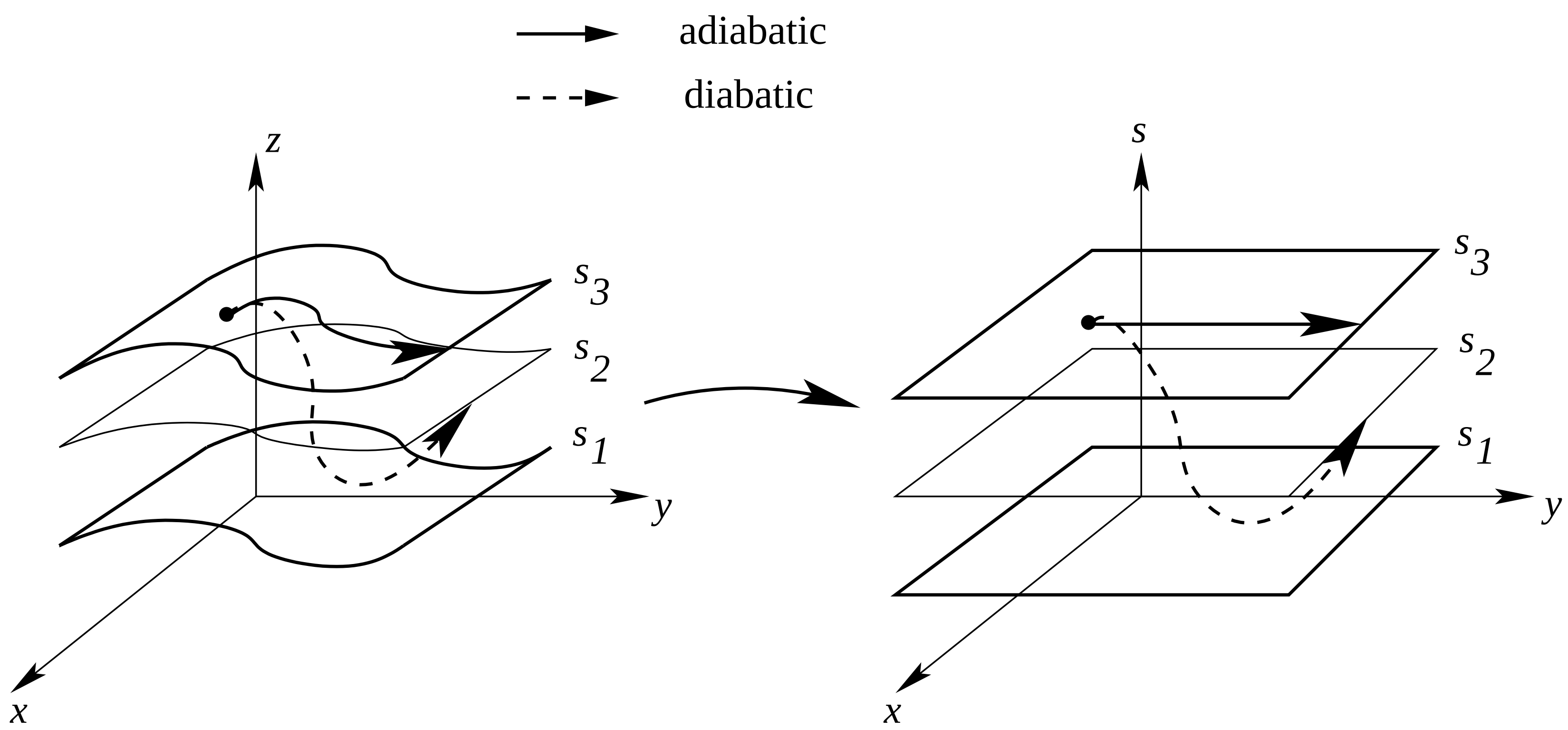
Open question:

How does numerical scheme with conservative core behave in presence of (weak) forcing and friction?

Eulerian Equations

- f- plane
- hydrostatic balance
- ideal gas
- statically stable atmosphere $\rightarrow$ isentropic coordinates (x, y, s)

Why isentropic coordinates?



Eulerian Equations in (x, y, s)

- momentum eq.

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} + f \boldsymbol{v}^{\perp} + \boldsymbol{\nabla} M = 0 \tag{1}$$

- continuity eq.

$$\frac{\partial \sigma}{\partial t} + \boldsymbol{\nabla} \cdot (\sigma \boldsymbol{v}) = 0 \tag{2}$$

- hydrostatic eq. for ideal gas

$$\sigma = -\frac{p_{00}}{g} \frac{\partial}{\partial s} \left(\left(\frac{\sigma}{\rho_{00} \frac{\partial Z}{\partial s}} \right)^{\frac{c_p}{c_v}} e^{\frac{s - s_{00}}{c_v}} \right) \tag{3}$$

- $Z = Z(\boldsymbol{x}, s, t)$
- $M = g\,Z + c_p\,T(\sigma, \frac{\partial Z}{\partial s})$

$\rightarrow$ Closed formulation for $\{\boldsymbol{v}, \sigma, Z\}$

Picture:

Flow can be thought of as being horizontal, where all layers are coupled though eq. (3), which is nonlinear elliptic equation in s only.

Boundaries

- periodicity in (x, y)
- Z given at bottom (B) and top (T)
- $Z_{B,T}(x, y)$ is isentropic, somewhat restricted...

How find a Hamiltonian numerical scheme?

Parcel formulation

(Bokhove & Oliver, 2006)

- mixed Eulerian-Lagrangian
- fluid is continuum of parcels labelled by $(\boldsymbol{a}, s)$ having horizontal positions $\boldsymbol{X}(t; \boldsymbol{a}, s)$.
  (initially, labels and positions coincide)
- parcels are moving in a Eulerian prescribed potential

Why?

$\rightarrow$ discretization of this continuum formulation will 'directly' lead to a discrete Hamiltonian scheme!

Parcel formulation: equations

Motion of distinguished parcel $(\boldsymbol{A}, S)$ given by

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \boldsymbol{V} = \boldsymbol{\nabla}_{\boldsymbol{V}} H, \tag{4a}$$

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = -f\,\boldsymbol{V}^{\perp} - \boldsymbol{\nabla}_{\boldsymbol{X}} M = -f\,\boldsymbol{\nabla}_{\boldsymbol{V}}^{\perp} H - \boldsymbol{\nabla}_{\boldsymbol{X}} H, \tag{4b}$$

with $(\boldsymbol{\nabla}_{\boldsymbol{X}}, \boldsymbol{\nabla}_{\boldsymbol{V}}) \equiv (\frac{\partial}{\partial \boldsymbol{X}}, \frac{\partial}{\partial \boldsymbol{V}})$.

H is the parcel Hamiltonian,

$$H(\boldsymbol{X}, \boldsymbol{V}, t) = \frac{1}{2}\boldsymbol{V}^2 + M(\boldsymbol{X}, t). \tag{5}$$

- $M(\boldsymbol{X}, t) \equiv M(\boldsymbol{x}, s, t)\big|_{(\boldsymbol{x},s)=(\boldsymbol{X},S)}$

- Remember: Eulerian M found from Eulerian Z and σ.
  $\rightarrow$ Z obtained from σ, but how to recover σ from parcel data?

- Mass conservation law can be rewritten as

$$\sigma(\mathbf{x}, s, t) = \int\int \sigma_0(\boldsymbol{a}, s)\, \delta(\boldsymbol{x} - \boldsymbol{X}(t; \boldsymbol{a}, s))\, d\boldsymbol{a}$$

  $\rightarrow$ this equation couples all parcels.

Horizontal discretization:

Hamiltonian Particle-Mesh Method (HPM)

(Frank et al., 2002)

- spatial discretization of parcel formulation
- exact conservation of discrete energy
- Eulerian grid allows for efficient smoothing

- includes a *symplectic* time integrator assuring preservation of phase space structure
  $\rightarrow$ asymptotic conservation of energy

...still need to discretize in s-direction.

Vertical discretization: **Finite elements** (FEM)

Why finite elements?

→ managed to find conservation of energy in the reduced finite element function space

Numerical example: Flow over bump

- nondimensionalized
- $f = 0$, 2D-set up (x, s)
- nonlinear
- stationary
  $\rightarrow$ fluxes independent of x
- bump in middle of domain
- onflow $u(s) = 2\,s \rightarrow$ supercritical state
- bottom and top are isentropic

$\Rightarrow$ Can our Lagrangian particle scheme reproduce a Eulerian steady state?

Numerical parameters:

- 400 x-cells, 6 particles per cell initially
- smoothing length $2\,\Delta x$
- quadratic FEM with 30 isentropes
- $\Delta t = 0.005$, $T = 4$

MOVIE

Conclusion:

- Found a Hamiltonian numerical scheme for the hydrostatic Euler equations based on
  - HPM in horizontal
  - FEM in vertical

Outlook:

- bottom-intersecting isentropes $\rightarrow$ how to implement for particles?
- formulate stabilizer to avoid isentropes to overturn $\sim$ fronts
- add forcing & friction

Questions ??

References

- Bokhove, O., Oliver, M. 2006: Parcel Eulerian-Lagrangian fluid dynamics of rotating geophysical flows. *Proc. R. Soc. Lond.* **A 462**: 2563-2573.
- Frank, J., Gottwald, G., Reich, S. 2002: A Hamiltonian particle-mesh method for the rotating shallow-water equations. *Lecture Notes in Computational Science and Engineering*, vol. **26**. Springer, Heidelberg, 131-142.