EFFECTIVE BOUNDARY CONDITION FOR WAVE REFLECTION OVER SLOWLY VARYING BATHYMETRY

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MOTIVATION

Tsunamis (tsu = harbour, nami = wave) are long water waves generated by:
- offshore earthquakes,
- explosive volcanism near the surface of the ocean,
- submarine slides, or
- a meteorite that hit the ocean.
MOTIVATION

■ In the open ocean: long wavelength (hundreds of kilometres) vs its height (± 0.5m)

■ Its height increase in the last 10-20m depth of water before the shore
MOTIVATION

Present day simulation tools:

- Cannot calculate the waveheight near the shore accurately enough:
  - The interaction of the incoming waves with reflected waves from the coast.
  - Computing the details of run-up and run-down of waves on the shore is computationally expensive.

- Use a fixed wall or transparent boundary condition at a zone before the shoreline to simplify the problem (TUNAMI N2, FUNWAVE, GEOWAVE, etc).
Aim:
Design boundary conditions that are able to calculate more accurate wave interactions near the shore without increasing the computational cost.
Basic idea:
1. At $x=B$, information of the incoming wave is measured in time (denote by $d(t)$).
3. Select the information that accounts for the reflected waves influx $I$ into the sea at $x=B$. 
INTRODUCTION

At time $t$, the resulting reflected wave will depend on the incoming wave for all previous time, i.e:

$$I = M(d(t))$$

depends on $d(\tau)$ for all $\tau < t$. 

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INTRODUCTION

The challenges in this schematic overview of mapping the incoming waves to the outgoing reflected waves are:

1. Defining the measurement of the property of incoming waves operator $d(t)$
   - Linear Shallow Water Equations
2. Making a theoretical model for the wave interaction at the shore $M(d)$.
   - Over slowly varying bathymetry
3. Including the reflected wave properties $I=M(d)$ in an influx boundary condition.
   - Linear Shallow Water Equations
4. Implementing the above analytic results numerically.
   - Two-dimensional Finite Element Method, in one-dimensional setting.
LINEAR SHALLOW WATER EQUATIONS

\[
\begin{align*}
\partial_t \eta &= -\partial_x [h \partial_x \phi] \quad (1a) \\
\partial_t \phi &= -g \eta \quad (1b)
\end{align*}
\]

with \( \eta \) represents the wave elevation, and \( \phi \) represents the velocity potential (the velocity is given by \( u = \nabla \phi \)).
WKB APPROXIMATION

For slowly varying velocity $c(\varepsilon x)$, the WKB approximation for right traveling waves:

$$\eta_0(x,t) = \frac{A}{\sqrt{c(\varepsilon x)}} F(\theta(x,t))$$

where $\theta$ satisfies the eiconal equation

$$\left(\partial_t \theta\right)^2 = c^2 \left(\partial_x \theta\right)^2.$$
REFLECTION WKB APPROXIMATION

\[ \eta(x, t) = \eta_0 + \eta_1 \]

\[ \rightarrow \text{use WKB approximation} \]

\[ \rightarrow \text{ignore higher order term} \]
REFLECTION WKB APPROXIMATION

Given the initial condition $\eta(x,0) = F(x)$ then $\eta_0(x,0) = \sqrt{c(x)} F(x) = \bar{F}(x)$.

Then we have the solutions:

$$\eta_0 = \bar{F}(y-t) \quad (3)$$

$$\eta_1 = \int_y^{y+t} \bar{F}(2\beta - (y+t))B(\beta)d\beta \quad (4)$$

For $\eta_0 = \sqrt{c}\eta_0$ and $\eta_1 = \sqrt{c}\eta_1$

and introducing time independent variable $y = y(x)$ such that $\partial_y = c\partial_x \Rightarrow y = \int_0^x \frac{d\zeta}{c(\zeta)}$
COMPARISON BETWEEN REFLECTION WKB APPROXIMATION AND NUMERICAL SOLUTIONS

Bathymetry profile:

\[ h(x) = \frac{h_0 - h_1}{2} \cos \left( \frac{\pi}{2w} \left( x - m + w \right) \right) + \frac{h_0 + h_1}{2}, \]

- \( h_0 = 1000m \) is the depth before the slowly varying bathymetry
- \( h_1 = 100m \) is the depth after the slowly varying bathymetry
- \( m \) is the middle of the slope
- \( w \) is the half width of the slope
COMPARISON BETWEEN REFLECTION WKB APPROXIMATION AND NUMERICAL SOLUTIONS

$\varepsilon = 1/10$

Error average: 0.0032 m

- Reflection WKB approximation
- Numerical solution of linear SWE
COMPARISON BETWEEN REFLECTION WKB APPROXIMATION AND NUMERICAL SOLUTIONS

\[ \varepsilon = \frac{1}{20} \]

Error average: 0.0019 m

- Reflection WKB approximation
- Numerical solution of linear SWE
COMPARISON BETWEEN REFLECTION WKB APPROXIMATION AND NUMERICAL SOLUTIONS

\[ \varepsilon = 1/30 \]

Error average: 0.0013 m

- Reflection WKB approximation
- Numerical solution of linear SWE
COMPARISON BETWEEN REFLECTION WKB APPROXIMATION AND NUMERICAL SOLUTIONS
EFFECTIVE BOUNDARY CONDITIONS OVER SLOWLY VARYING BATHYMETRY

\[ d(t) = F\left(\frac{B}{c_B} + t\right) \]

\[ \eta(x,0) = F(x) \]

Influx reflected waves I:

\[ \partial_t \phi + c_B \partial_x \phi = -2g \eta_1(B,t) \]

Model M: Reflection WKB approximation
SIMULATIONS
B=20km, L=100km

CPU time for numerical calculation in the whole domain: 120s
CPU time to calculate the analytical solution of the reflection wave: 8.60s
CPU time for calculation with EBC: 82s

- simulation with EBC
- simulation in the whole domain
BATHYMETRY OF LAMPUNG, SOUTH OF SUMATRA

Bathymetry of Lampung

Sumatra

JAVA

Ridge

longitude[deg]

latitude[deg]

depth [m]
BATHYMETRY OF PANGANDARAN, SOUTH OF JAVA
CONCLUSIONS

- The Effective Boundary Conditions (EBCs) over slowly varying bathymetry have been derived, implemented, and compared with the numerical solutions.
  - The comparisons show there is some error that is expected because we used second order WKB approximation.
  - When the slope is steeper, the error is larger.

- By using this EBC, the computational time can be reduced.

- The Reflection WKB approximation also improves the WKB approximation for the wave that propagates over the slowly varying bathymetry.
FUTURE WORK

- Deriving EBC when there is run-up an run-down on the shore (adding friction, etc.).

- Improving the propagation wave model, e.g. using Variational Boussinesq Model, non linear model.

- Go to two dimensional case.
THANK YOU