A Hamiltonian Numerical Scheme for Large Scale Geophysical Fluid Systems

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Outline

1. Introduction & Motivation

2. Hamiltonian parcel formulation for shallow water equations (SWE)

3. Hamiltonian particle-mesh method

4. Numerical example

5. Generalization to adiabatic atmosphere

6. Conclusions & Outlook
**Given:**
In the limit of no forcing and dissipation, our climate system is governed by conservation laws:

\[
\frac{\partial T}{\partial t} + \nabla \cdot F = 0
\]

- \( T \) = mass, momentum, energy and vorticity (barotropic fluid);
  and \( F \) its associated flux,

- system can be rewritten in Hamiltonian way,

- it has a special phase space structure: symplecticity

**However:**
Climate system is weakly dissipative.
So why bother about Hamiltonian systems?
Hypothesis:

Numerical methods that reproduce the correct flow structure in limit of no forcing and dissipation provide better climate predictions.

Arguments??

+ Promising results for symplectic time integration for ensemble of dissipatively perturbed low-order models (e.g., Cotter & Reich, 2003).

+ Since that symplectic time integrators recover the limiting behavior correctly, they allow for accurate long time integrations of such models (Hairer et al., 2002).
**Phase space structure ??**

Consider 1-particle mechanical system

\[
\ddot{x} = -\frac{dV}{dx} \quad \Leftrightarrow \quad \dot{x} = v \quad \& \quad \dot{v} = -\frac{dV}{dx}
\]

Phase space: collection \(\{x, v\} \in \mathbb{R}^2\)

Phase fluid \(\Phi = (\dot{x}, \dot{v})\) is divergence free:

\[
\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{v}}{\partial v} = 0
\]
Introduction & Motivation

V

X

t=0

t=T
Our starting point: **no** forcing and friction

Tasks:

- Find Hamiltonian description of fluid system.
- Find Hamiltonian discretization.

Point:

No general method for reducing noncanonical Hamiltonian PDEs to a finite-dim Hamiltonian system.

→ can’t use **Eulerian** fluid equations

→ adopt **Lagrangian** point of view

→ use discretization based on Hamiltonian **parcel formulation** (Bokhove & Oliver, 2006).
Hamiltonian parcel formulation for SWE

Shallow Water Equations:

Eulerian description

\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + f \mathbf{v}^\perp + g \nabla h = 0
\]

\[
\frac{\partial h}{\partial t} + \nabla \cdot (h \mathbf{v}) = 0
\]

- \( \mathbf{v} = (u, v) \), \( \mathbf{v}^\perp = (-v, u) \)

- \( h(x, y, t) \rightarrow \) height of the fluid free surface

- \( f \rightarrow \) Coriolis parameter: effect of earth rotation
Hamiltonian parcel formulation for SWE

Hamiltonian description of fluid system

steps:

1. view fluid as continuum of fluid parcels
2. each parcel has fixed (infinitesimal) mass

Parcel formulation

⇔

what are the parcel trajectories for the SWE?
Hamiltonian parcel formulation for SWE

Parcel’s motion given by

\[
\frac{dX}{dt} = \nabla_v H = V, \tag{1a}
\]
\[
\frac{dV}{dt} = -f \nabla_v^\perp H - \nabla_x H = -f V^\perp - g \nabla_x h, \tag{1b}
\]

with \((\nabla_x, \nabla_v) \equiv (\frac{\partial}{\partial X}, \frac{\partial}{\partial V}).\)

The parcel’s Hamiltonian reads

\[
H(X, V, t) = \frac{1}{2} V^2 + g h(X, t). \tag{2}
\]
Crux I: how to obtain the fluid depth $h$ from parcel data?

\[ h(X, t) = \int h(a, 0) \delta((X) - \chi^t(a)) \, da. \] (3)

Indeed: integrand nonzero $\Leftrightarrow \chi^t(a) = \chi^t(A) \equiv X$.

Crux II: we view (3) as Eulerian function $h(x, t)$, evaluated at $x = X$. Why??
Hamiltonian discretization:

**The Hamiltonian Particle-Mesh Method (HPM)**
(Frank et al., 2002)

procedure simple: \( \int \) parcels \( \Rightarrow \sum \) particles
The algorithm:

i) Fluid decomposed in $N$ particles.

ii) Lagrangian step:
   Apply symplectic time integrator to move the particles.

iii) Eulerian steps:
   a. Fixed grid $(x_i, y_j)$.
   b. Redistribution of particles induces new density field. Density on grid found from discretization of (3).
   c. Interpolate grid values $h_{ij}(t)$ back to $\bar{h}(x, y, t)$ using appropriate interpolation function.
Smoothing:

i) Weak point in our discretization is the truncated form of (3)

ii) We like to apply smoothing to the particle depth values.

iii) If we introduce a Eulerian grid, we can compute the smoothing using FFT.
Numerical example: Burger’s profile

Assume

• 1-dimensional flow, $f = 0$.
• $u(x,t) + 2\sqrt{gh(x,t)} = K$, with $K$ constant.

Then the SWE reduces to Burgers’ equation

$$\frac{\partial q}{\partial t} + q\frac{\partial q}{\partial x} = 0$$

for $q(x,t) = K - 3\sqrt{gh(x,t)}$.

→ **Analytical** solution implicitly defined by $q(x,t) = q_0(x - q(x,t)t)$.
→ Wave breaking after certain time.
**Numerical example**

**Movie**

**Input**

- Initial profile: \( q_0 = \sin(x) \) with \( K = 3 \) on domain \([0, 2\pi]\).
  \( \rightarrow \) Wave breaking at \( t = 1 \). Take 200 time steps.

- 100 gridpoints, 2 particles per cell initially.

- **Second** order symplectic time integrator.

- Smoothing operator to avoid non-smooth interpolation of grid values \( h_{ij} \).

**Output**

- Nearly **second** order spatial convergence.

- Exponentially small drift in total energy.
Generalization to adiabatic atmosphere

Model assumptions:

1) fluid on rotating plane,
2) hydrostatic balance,
3) reversible thermodynamics $\iff$ entropy conserved per fluid parcel,
4) statically stable atmosphere.

Denote specific entropy by $s$.

$\rightarrow$ statically stable conditions if $\frac{\partial s}{\partial z} > 0$. 
3) + 4) allow for isentropic frame of reference: \((x, y, z) \rightarrow (x, y, s)\).

- Since entropy is materially conserved, the flow becomes horizontal within this frame: \(\mathbf{v} = (u, v, 0)^T\), \(\nabla \equiv (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0)^T\).
The equations

- momentum eq.
  \[ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + f \mathbf{v}^\perp + \nabla M = 0 \]  
  \[ (4) \]

- continuity eq.
  \[ \frac{\partial \sigma}{\partial t} + \nabla \cdot (\sigma \mathbf{v}) = 0 \]  
  \[ (5) \]

- hydrostatic eq. for ideal gas
  \[ \sigma = -\frac{p_{00}}{g} \frac{\partial}{\partial s} \left( \left( \frac{\sigma}{\rho_{00}} \frac{\partial Z}{\partial s} \right) \frac{c_p}{c_v} e^{\frac{s-s_{00}}{c_v}} \right) \]  
  \[ (6) \]
4. Atmospheric equations

- $\sigma$ is pseudodensity,
- $M = g Z + c_p T(\sigma, \frac{\partial Z}{\partial s})$ is the Montgomery potential.

$\rightarrow$ Closed formulation for $\{v, \sigma, Z\}$.

- $v = v(x, s, t)$
  $\sigma = \sigma(x, s, t)$
  $Z = Z(x, s, t)$

**Picture:**
Flow can be thought of as being horizontal, where all layers are coupled though eq. (6), which is nonlinear elliptic equation in $s$ only.
4. Atmospheric equations

Parcel formulation for adiabatic model

Changes w.r.t. parcel formulation SWE:

- $h \rightarrow M$ in momentum equation and Hamiltonian,
- $\rho \rightarrow \sigma$ in continuity equation,
- New equation (6) for the coupling $\sigma$ to $M$.

Conclusion:

In isentropic coordinates, the adiabatic atmospheric flow forms a continuum stack of 2D SWE type of flows coupled via a 1D elliptic inversion.
Outlook

- We have found a Hamiltonian discretization for the adiabatic atmosphere (for simplified boundary conditions), based on HPM and a Finite Element discretization in the $s$-direction.

Outlook

- Extend to more general boundary conditions.
- Find way to account for unresolved but important dynamics, like gravity waves.
- Extend to a spherical model.
- **Forcing & friction.**
Questions ??
References


