Option Pricing and Hedging via Risk Measures

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PhD-TW Colloquium, 09.04.2009
Outline

- Modelling in finance: stocks, bonds, options and pricing of options
- Risk measures: properties and representation
- Pricing via risk measures
**Financial market**

Let the filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})\) be given. The market consists of two price processes: stocks \((S_t)_{t\in[0,T]}\) and bonds \((B_t)_{t\in[0,T]}\):

\[
\frac{dS_t}{S_t} = \mu dt + dM_t, \quad \frac{dB_t}{B_t} = rdt,
\]

for some martingale \((M_t)_{t\in[0,T]}\), a stochastic process. The associated self-financing portfolio value process \(X\) is defined as

\[
\frac{dX^\pi,x_t}{X^\pi,x_t} = \pi_t \frac{dS_t}{S_t} + (1 - \pi_t) \frac{dB_t}{B_t}, \quad X^\pi,x_0 = x,
\]

for a trading strategy \((\pi_t)_{t\in[0,T]} \in \Pi\). This means: perc. change in \(X = \text{perc. invested in } S \times \text{perc. change in } S + \text{perc. invested in } B \times \text{perc. change in } B\).
European option

A European option is the right, but not the obligation to buy or sell a particular stock $S$ at a certain time $T$ for a certain price $K$ and as such can be modelled as follows:

(i) An expiration date $T \in (0, \infty)$.

(ii) A terminal payoff $\Phi(S_T)$, for example $(S_T - K)^+$ for the call, where $K \geq 0$ is called the strike price of the option.
Complete market

A market is complete if every option is hedgeable, this means for every $\Phi$ there exists a self-financing portfolio $\pi$ and an initial price $x$ such that

$$\mathbb{P}(X^\pi_T = \Phi(S_T)) = 1.$$ 

The price of the option is the initial price $x$. 
Incomplete market

In incomplete markets some options are not hedgeable! A conservative approach to hedging is to look for a self-financing strategy $\pi$ such as to remain always on the safe side

$$\mathbb{P}(X_T^{\pi,x} \geq \Phi(S_T)) = 1.$$ 

Such a strategy is called a superhedging strategy. The cost of the cheapest superhedging strategy is the price of the option

$$\inf\{x : \exists \pi \in \Pi, \mathbb{P}(X_T^{\pi,x} \geq \Phi(S_T)) = 1\}.$$ 

For a call option $(S_T - K)^+$ the superhedging price is given by $S_0$, which is much too high.
Risk measure

A risk measure is a mapping from stochastic variables (profit/loss) to the real numbers

$$\rho : L^1(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{R}.$$  

When negative, i.e. $\rho(X) \leq 0$, then the risk is acceptable. When positive, $\rho(X)$ is the minimum extra cash the agent has to add to her portfolio $X$ to make the risk acceptable.
Convex risk measure

A mapping $\rho : L^1(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ is called convex risk measure if
$\rho(0) = 0$ and it satisfies the following conditions for all $X, X_n, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, $c \in \mathbb{R}$, $\lambda \in [0, 1]$:

Translation invariance: $\rho(X + c) = \rho(X) - c$.

Monotonicity: $X \leq Y$ implies $\rho(X) \geq \rho(Y)$.

Convexity: $\rho(\lambda X + (1 - \lambda) Y) \leq \lambda \rho(X) + (1 - \lambda) \rho(Y)$.

Fatou property: If $(X_n)_{n \in \mathbb{N}}$ is bounded, $X_n \to X$, then
$\rho(X) \leq \lim \inf_{n \to \infty} \rho(X_n)$.

If, in addition:

Positive homogeneity: $\rho(\Lambda X) = \Lambda \rho(X)$, for $\Lambda \geq 0$,
then $\rho$ is called a coherent risk measure.
Dual representation

By the Fenchel-Moreau Theorem, every convex lower semicontinuous functional and thus certainly every convex risk measure $\rho$ on $L^p$ has a representation of the form

$$\rho(X) = \sup\{\mathbb{E}(XZ) - \rho^*(Z) : Z \in L^q\},$$

(1)

where $\rho^*(Z) = \sup\{\mathbb{E}(XZ) - \rho(X) : X \in L^p\}$ is the conjugate of $\rho$. We call (1) dual representation.
Robust representation

If \( X \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \), the dual representation looks like

\[
\rho(X) = \sup \{ \mathbb{E}_Q(-X) - \alpha(Q) : Q \in \mathcal{P} \},
\]

(2)

where \( \mathcal{P} := \{ Q \text{ probability measures on } (\Omega, \mathcal{F}) | Q \ll \mathbb{P} \text{ on } \mathcal{F} \} \) and \( \alpha(Q) = \rho^*(-\frac{dQ}{d\mathbb{P}}) \). Properties of \( \rho \) follow from the dual representation

- \( \rho \) is translation invariant, iff all \( Z \in \text{dom}(\rho^*) \) satisfy \( \mathbb{E}(Z) = -1 \).
- \( \rho \) is monotone, iff all \( Z \in \text{dom}(\rho^*) \) satisfy \( Z \leq 0 \).
- \( \rho \) is positive homogeneous, iff \( \alpha(\cdot) \) takes only the values 0 and \( +\infty \). In this case, we can rewrite (2) as

\[
\rho(X) = \sup \{ \mathbb{E}_Q(-X) : Q \in \mathcal{Q} \},
\]

with \( \mathcal{Q} := \{ Q \in \mathcal{P} | \alpha(Q) = 0 \} \).
Conditional risk measure

One can define conditional risk measures similar to the definition of conditional expectation. The *conditional risk measure* is a mapping

$$\rho_t : L^1(\Omega, \mathcal{F}_T, \mathbb{P}) \to L^0(\Omega, \mathcal{F}_t, \mathbb{P}).$$

An example of a conditional coherent risk measure is expected shortfall:

$$\rho_t(X_T) = \mathbb{E}(-\min\{X_T, 0\} | \mathcal{F}_t).$$
Pricing and hedging with vanishing risk

The cost of superhedging of $\Phi(S_T)$ is too high. A risk adjusted price for $\Phi(S_T)$ should be such that it allows for hedging of $\Phi(S_T)$ with vanishing risk:

$$\inf\{x : \exists \pi \in \Pi, \rho_t(X_t^{\pi,x} - \Phi(S_T)) \leq 0, \forall t \in [0, T]\}.$$ 

Vanishing risk does not mean exact hedging, but a self-financing hedge such that the resulting replication error has vanishing risk w.r.t. to a certain risk measure.
Capital reserve

We propose to introduce an extra bank account $Z$, which serves as a capital reserve. The capital reserve has an interest rate smaller than the risk free rate and borrowing from $Z$ is not allowed. The price processes $S$, $B$ and $Z$ are given by

$$\frac{dS_t}{S_t} = \mu dt + dM_t, \quad \frac{dB_t}{B_t} = r dt, \quad \frac{dZ_t}{Z_t} = \tilde{r} dt,$$

for some martingale $M$ with $0 \leq \tilde{r} < r$. 
Pricing and hedging with bounded risk

Define the portfolio value process $X$ as

$$
\frac{dX^{\pi,x}_t}{X^{\pi,x}_t} = \pi_t \frac{dS_t}{S_t} + (1 - \pi_t) \frac{dB_t}{B_t} + \frac{dZ_t}{X^{\pi,x}_t}, \quad X^{\pi,x}_0 = x,
$$

for a trading strategy $(\pi_t)_{t \in [0,T]} \in \Pi$. This means at time 0 the agent invests a fraction of her wealth $c$ in $Z$ and the remaining part $x - c$ in $S$ and $B$.

The price of the option is defined by

$$
\inf \{ x : \exists \pi \in \Pi, \rho_t(X^{\pi,x}_T - \Phi(S_T)) \leq ce^{\tilde{r}t}, \forall t \in [0, T] \}.
$$

The capital reserve is the amount of money the agent has to put aside to cover her exposure to risk.
Pricing and hedging with dynamic risk

Let $\pi^S$ be the proportion of wealth invested in stocks, $\pi^B(\pi^Z)$ the proportion invested in bonds (capital reserve) with $\pi^S_t + \pi^B_t + \pi^Z_t = 1$ and $\pi^Z_t \geq 0$ for all $t \in [0, T]$. Given a trading strategy $\pi \in \Pi$, the value process of the portfolio is

$$\frac{dX_{t}^{\pi,X}}{X_{t}^{\pi,X}} = \pi^S_t \frac{dS_t}{S_t} + \pi^B_t \frac{dB_t}{B_t} + \pi^Z_t \frac{dZ_t}{Z_t}, \quad X_{0}^{\pi,X} = x,$$

with $\pi = (\pi^S, \pi^B, \pi^Z)$. The price of the option is given by

$$\inf\{x : \exists \pi \in \Pi, \rho_t(X_{T}^{\pi,X} - \Phi(S_T)) \leq \pi^Z_t X_{t}^{\pi,X}, \forall t \in [0, T]\}.$$