1 Stabilization of a Double Pendulum

The purpose of this exercise is to illustrate the full extent of the theory developed in Chapters 9 and 10. The exercise uses many of the concepts introduced in this book (modeling, controllability, observability, stability, pole placement, observers, feedback compensation). This exercise requires extensive use of computer aids: MAPLE for formula manipulation, and MATLAB and SIMULINK for control system design and numerical simulation.

1.1 Modeling

We study the stabilization of a double pendulum mounted on a movable cart. The relevant geometry is shown in Figure 1. It is assumed that the motion takes place in a vertical plane. The

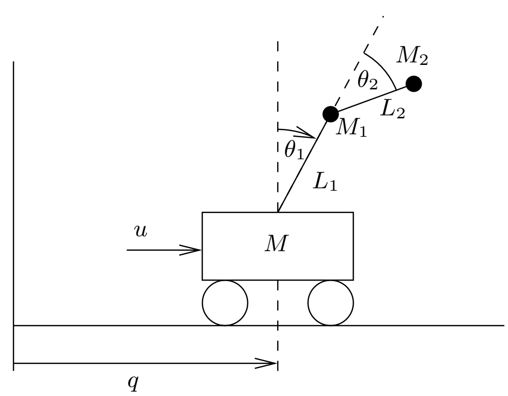

Figure 1: A double pendulum on a cart.

significance of the system parameters is as follows:

M :	mass of the cart
M_1:	mass of the first pendulum
M_2:	mass of the second pendulum
L_1:	length of the first pendulum
L_2:	length of the second pendulum

The cart and the pendula are all assumed to be point masses, with the masses of the pendula concentrated at the top. It is instructive, however, to consider how the equations would change if the masses of the pendula are uniformly distributed along the bars.

The significance of the system variables is as follows:

u:	the external force on the cart
q:	the position of the cart
θ_1:	the inclination angle of the first pendulum
θ_2:	the inclination angle of the second pendulum

For the output y we take the 3-vector consisting of the horizontal positions of the cart and of the masses at the top of the pendula.

The purpose of this exercise is to develop and test a control law that holds the cart at a particular position with the pendula in upright position. We assume that all three components of the output y are measured and that the force u is the control input. Our first order of business is to find the dynamical relation between u and y.

The dynamical equations of this system are given by

$$\begin{aligned}
&-(L_1M_1+L_1M_2)(\tfrac{d\theta_1}{dt})^2\sin\theta_1 - L_2M_2\tfrac{d\theta_1}{dt}\ \tfrac{d\theta_2}{dt}\sin(\theta_1+\theta_2)\\
&-L_2M_2(\tfrac{d\theta_2}{dt})^2\sin(\theta_1+\theta_2)+(M+M_1+M_2)\tfrac{d^2q}{dt^2}\\
&+(L_1M_1+L_1M_2)\tfrac{d^2\theta_1}{dt^2}\cos\theta_1+L_2M_2\tfrac{d^2\theta_2}{dt^2}\cos(\theta_1+\theta_2) \quad = \quad u,\\
&-gL_1(M_1+M_2)\sin\theta_1-gL_2M_2\sin(\theta_1+\theta_2)\\
&+L_2M_2\tfrac{dq}{dt}\ \tfrac{d\theta_2}{dt}\sin(\theta_1+\theta_2)-L_1L_2M_2(\tfrac{d\theta_2}{dt})^2\sin\theta_2\\
&+(L_1M_1+L_1M_2)\tfrac{d^2q}{dt^2}\cos\theta_1+L_1^2(M_1+M_2)\tfrac{d^2\theta_1}{dt^2}\\
&+L_1L_2M_2\tfrac{d^2\theta_2}{dt^2}\cos\theta_2 \quad = \quad 0,\\
&-gL_2M_2\sin(\theta_1+\theta_2)-L_2M_2\tfrac{dq}{dt}\ \tfrac{d\theta_1}{dt}\sin(\theta_1+\theta_2)\\
&+L_2M_2\tfrac{d^2q}{dt^2}\cos(\theta_1+\theta_2)+L_1L_2M_2\tfrac{d^2\theta_1}{dt^2}\cos\theta_2+L_2^2M_2\tfrac{d^2\theta_2}{dt^2} \quad = \quad 0.
\end{aligned}$$

Completed with the output equation

$$y=\begin{bmatrix} q \\ q+L_1\sin\theta_1 \\ q+L_1\sin\theta_1+L_2\sin(\theta_1+\theta_2)\end{bmatrix}, \tag{2}$$

we obtain a full system of equations relating the input to the output.

1.2 Linearization

Task 1 Prove that $u^*=0, q^*=0, \theta_1^*=0, \theta_2^*=0, y^*=0$ is an equilibrium solution. Explain physically that this is as expected. Do you see other equilibria?

Task 2 Introduce as state variables $x_1=q, x_2=\theta_1, x_3=\theta_2, x_4=\dot q, x_5=\dot\theta_1, x_6=\dot\theta_2$. Derive the input/state/output equations; i.e., write the equations in the form

$$\frac{dx}{dt}=f(x,u),\ \ y=h(x). \tag{3}$$

Note that the dynamical equations have the form $V(x)[\dot x_4\ \dot x_5\ \dot x_6]^T=W(x,u)$. We see that $[\dot x_4\ \dot x_5\ \dot x_6]^T$ can be written in the form $V(x)^{-1}W(x,u)$.

Task 3 Use MAPLE to linearize the nonlinear input/state/output equations about equilibrium point $\bar x=0, \bar u=0$. You should obtain the following equations:

$$\frac{d\Delta x}{dt}=A\Delta x+B\Delta u, \qquad \Delta y=C\Delta x, \tag{4}$$

with

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -\frac{g(L_1(M_1+M_2)+L_2M_2)}{L_1M} & -\frac{gL_2M_2}{L_1M} & 0 & 0 & 0 \\ 0 & \frac{g(L_1M_1(M+M_1+M_2)+L_2M_2(M+M_1))}{L_1^2MM_1} & \frac{gM_2(-L_1M+L_2(M+M_1))}{L_1^2MM_1} & 0 & 0 & 0 \\ 0 & -\frac{gM_2}{L_1M_1} & \frac{g(L_1(M_1+M_2)-L_2M_2)}{L_1L_2M_1} & 0 & 0 & 0 \end{bmatrix}, \tag{5a}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{M} \\ -\frac{1}{L_1M} \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & L_1 & 0 & 0 & 0 & 0 \\ 1 & L_1+L_2 & L_2 & 0 & 0 & 0 \end{bmatrix}. \tag{5b}$$

Assume henceforth the following reasonable choices for the system parameters:
$M = 100$ kg, $M_1 = 10$ kg, $M_2 = 10$ kg, $L_1 = 2$ m, $L_2 = 1$ m, $g = 10$ ms^{-2}

Task 4 Compute A, B and C.

1.3 Analysis

Task 5 Is the equilibrium a stable, asymptotically stable or unstable equilibrium of the nonlinear system? Give a physical explanation.

Task 6 Calculate the eigenvalues of A. Is the linearized system stable, asymptotically stable or unstable ?

Task 7 Show that the system is controllable and observable.

Task 8 Plot the Bode diagrams, with u as input and y as output. Note that you should have three diagrams, one for each of the output components.
Hint: use MATLAB command `bode`.

1.4 Implementation of the nonlinear system

Make a MATLAB M-file `dx456.m` that contains the following lines

```
function [xp] = dx456(x1,x2,x3,x4,x5,x6,u)
  c2  = cos(x2);      s2  = sin(x2);
  c3  = cos(x3);      s3  = sin(x3);
  c23 = cos(x2+x3);   s23 = sin(x2+x3);
  V = [
       120      40*c2   10*c23
        40*c2   80      20*c3
        10*c23  20*c3   10
      ];
  W = [
          10*(x5+x6)*x6*s23 + 40*x5^2*s2 + u
       ( 100 - 10*x4*x6)*s23 +     400*s2 + 20*x6^2*s3
```

```
      ( 100 + 10*x4*x5)*s23
     ];
xp = inv(V)*W;
```

Task 9 Check (with MAPLE) that this function returns $[\dot{x}_4\ \dot{x}_5\ \dot{x}_6]^T$

Task 10 Make a SIMULINK model of the nonlinear system. You can use the model below as a starting point. The parameters of the block `dx456` are:
`Matlab function:  dx456(u(1),u(2),u(3),u(4),u(5),u(6),u(7))`
`Output dimensions:  3`

SIMULINK tip : use `mux/demux` to combine/extract variables.

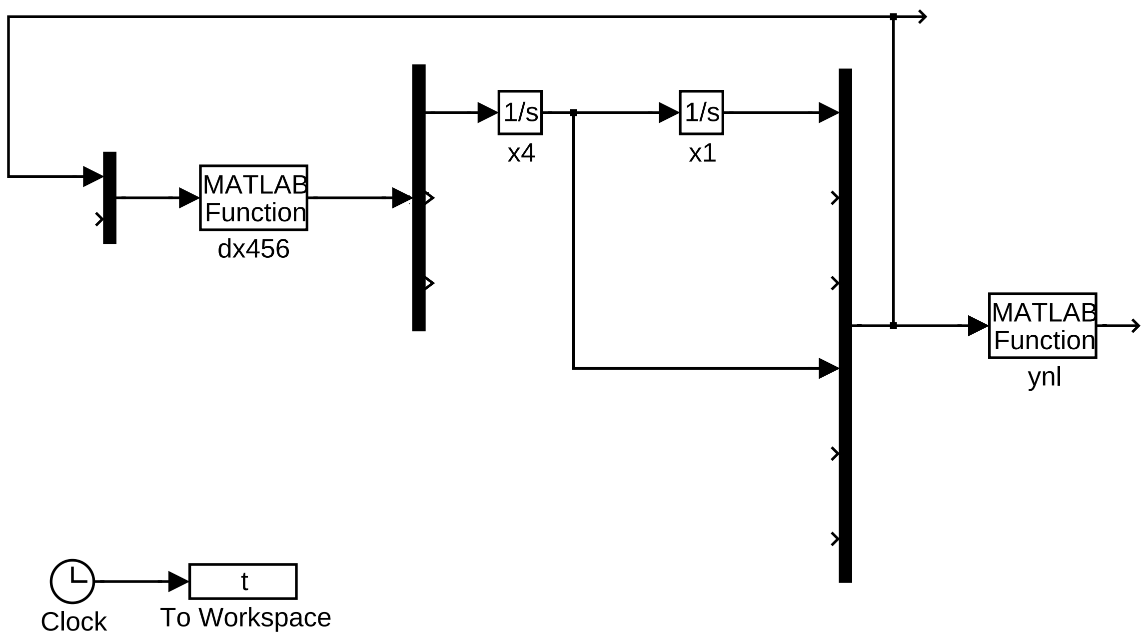

Figure 2: Simulink model

Task 11 Simulate the behavior of (x_1, x_2, x_3). Take input $u = 0$ and initial condition $x_1(0) = 0, x_2(0) = 2.5, x_3(0) = 0, x_4(0) = 0, x_5(0) = 0, x_6(0) = 0$. Take $t_{final} = 10$. Use the (fixed step) ode-4 (Runge-Kutta) integration method. Take step-size 0.02.
MATLAB tips :
you can use MATLAB command `sim` to simulate your model.
`plot(t,x(:,1:3))` gives the behavior of the first three variables of x.
you can use script-files (with extenstion `m`) to execute several MATLAB commands at once.

Task 12 What do you see? Give a physical explanation.

1.5 Stabilization

We first stabilize the linear system using state feedback. We can do this because the linearized system is controllable. The system is sixth order, the control is a scalar. Thus we have to choose six eigenvalues, $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$, in the left half plane and compute the six components of the feedback gain N such that the closed loop system matrix $A + BN$ has the desired eigenvalues.

Task 13 Use MATLAB to compute the state feedback gain N with the following λs: $-7.5 \pm 0.3i, -6.5 \pm 0.9i, -3.3 \pm 2.3i$. Don't use command `place`
MATLAB tip: use `ctrb`, `poly` and `polyvalm`.

Task 14 Make a SIMULINK block diagram of the linear system. Note that you will need the state Δx to connect the state feedback controller.
SIMULINK tip: you can use the variables that you define in MATLAB, in your model.

Task 15 Combine the state feedback controller and the linear system. (Use `Matrix gain`). Make a simulation of the closed loop system with the following initial conditions: $\Delta x_1(0) = -0.5, \Delta x_2(0) = 0, \Delta x_3(0) = 0, \Delta x_4(0) = 0, \Delta x_5(0) = 0, \Delta x_6(0) = 0$. This corresponds to making a maneuver: the cart is moved from one equilibrium position to the desired one, with the cart at the origin. Plot the transient response of $(\Delta x_1,\ \Delta x_2,\ \Delta x_3)$. Take $t_{final} = 5$ and step size 0.01.

Task 16 What do you see? Give an explanation

Task 17 **(optional)** Make simulations with different λs. Compare the results (in terms of overshoot and settling time).

Task 18 Make a SIMULINK block diagram with the state feedback controller and your nonlinear system. Make a simulation with $x(0) = (-0.5, 0, 0, 0, 0, 0)$. Plot the transient responses x_1, x_2, x_3

Task 19 Give a short explanation.

Note that you obtained a good transient response notwithstanding a rather high initial disturbance. Observe in particular the interesting small time behavior of $x_1 = q$.

1.6 Compensator

An observer of the linear system is given by

$$\frac{d\hat{x}}{dt} = (A - LC)\hat{x} + Bu + Ly \tag{6}$$

Task 20 Use MATLAB command `place` to compute a matrix L such that the eigenvalues of the error dynamics matrix $A - LC$ are: $-10, -10, -5, -3, -1, -1$.
MATLAB tip: `transpose` transposes a matrix.

Combine the state feedback controller and the observer in order to obtain the following compensator:

$$\frac{d\hat{x}}{dt} = (A - LC + BN)\hat{x} + Ly \tag{7}$$

$$u = N\hat{x} \tag{8}$$

Task 21 Make a SIMULINK model of the calculated compensator and the linearized system.

The error e is the difference between the real state and the estimated state.

Task 22 Test the controller by plotting the transient responses of x_1, x_2, x_3 and the error e_1, e_2, e_3 for $\Delta x(0) = (-0.5, 0, 0, 0, 0, 0)$ and $\hat{x}(0) = \Delta x(0)$ and for $\Delta x(0) = (-0.5, 0, 0, 0, 0, 0)$ and $\hat{x}(0) = (-0.4, 0, 0, 0, 0, 0)$.

Task 23 What do you see? Give an explanation

Task 24 Make a SIMULINK model of the calculated compensator and the nonlinear system.

Task 25 Use the same initial conditions for a simulation of this nonlinear model: $x_1(0) = -0.5, x_2(0) = 0, x_3(0) = 0, x_4(0) = 0, x_5(0) = 0, x_6(0) = 0$ and $\hat{x}_1(0) = -0.4, \hat{x}_2(0) = 0, \hat{x}_3(0) = 0, \hat{x}_4(0) = 0, \hat{x}_5(0) = 0, \hat{x}_6(0) = 0$

Task 26 Give a short explanation

1.7 Your report

Include all (relevant) MAPLE calculations. Include MATLAB script-files and relevant function files. Include the Simulink block diagrams of the compensated linear system and of the compensated nonlinear system. What is the contents of the blocks ?

Good luck.