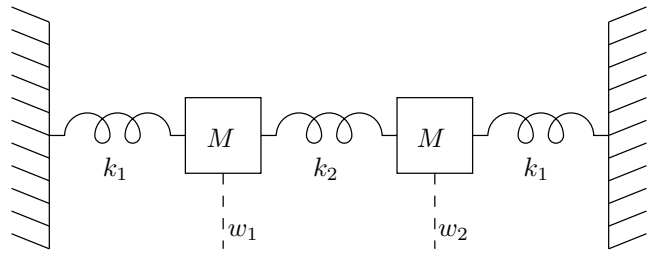
5 Dynamics of coupled masses

The purpose of this exercise is to study some interesting oscillatory phenomena. Consider the following mass-spring system:



We assume that there is no influence by gravitational or frictional forces. The two masses are taken to be unity. When both masses are in their respective equilibrium positions, the three springs are neither compressed nor stretched. The variables w_1 and w_2 are the displacements from the equilibrium position of the masses.

Tasks

1. Using Newton's second law, that is the sum of the forces = mass × acceleration ($\sum F = m \times a$), derive the force balance for both masses.

2. Rewrite the two force balances to $\frac{dw}{dt} = Aw$, with $w = [w_1, \frac{dw_1}{dt}, w_2, \frac{dw_2}{dt}]^{\mathrm{T}}$ and A a 4×4-matrix with constant coefficients.

3. Determine the characteristic polynomial and the eigenvalues of the matrix A. MAPLE could serve as a useful tool. Be careful: $k_1, k_2 > 0$

4. Derive the general form of the trajectories in the behavior of this mass-spring system, that is the general solution to $\frac{dw}{dt} = Aw$ (See chapter 2 of the IWS-syllabus or Chapter 3 of the book: take an arbitrary initial condition $w(0)$). MAPLE is useful here. Write the solution in trigonometric form as follows:

$$\begin{aligned} w_1(t) &= \alpha \cos\sqrt{k_1}t + \beta \sin\sqrt{k_1}t + \gamma \cos\sqrt{k_1+2k_2}t + \delta \sin\sqrt{k_1+2k_2}t \\ w_2(t) &= \alpha \cos\sqrt{k_1}t + \beta \sin\sqrt{k_1}t - \gamma \cos\sqrt{k_1+2k_2}t - \delta \sin\sqrt{k_1+2k_2}t \end{aligned}$$

5. Take $k_1 = 36$ and $k_2 = 4$. Physically this means that the masses are connected to the walls by means of relatively strong springs, whereas their mutual spring connection is rather weak. Make a block diagram of this system in the SIMULINK-extension of MATLAB. Hint: Use the block **State − Space**. We use this block diagram throughout the rest of this exercise.

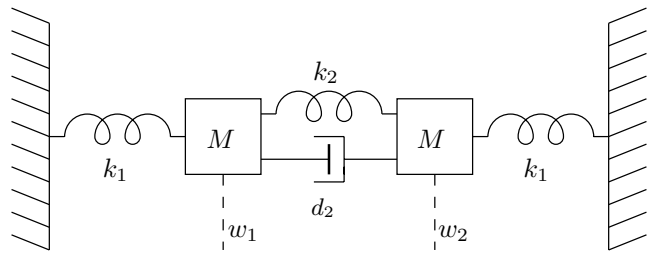
   Assume that at time $t = 0$ both masses have velocity zero, with the first mass displaced from its equilibrium by one unit, and the second mass in its equilibrium. Simulate the behavior of (w_1, w_2) with this initial condition using MATLAB and your SIMULINK-blockdiagram. Let t run from 0 to 100 with step time 0.01. For the actual simulation use 'rk45' instead of 'euler' as integration method. Observe that it appears as if the periodic motion of the two masses is periodically exchanged between them.

6. Use the formula $\cos p + \cos q = 2\cos\frac{p-q}{2}\cos\frac{p+q}{2}$ to explain that the behavior of w_1 and w_2 in task 5 can be seen as a fast oscillation modulated by a slowly oscillating amplitude. Use the goniometric form given in task 4 to do this: the values for α, β, γ and δ follow from the initial conditions given in task 5. Determine the slow and the fast frequencies.

7. Show for the solution derived above that the slowly oscillating amplitudes of w_1 and w_2 are in *antiphase*. Explain what you mean by this.

8. Now take $k_1 = 4$ and $k_2 = 36$. This corresponds to the situation where the two masses are connected to each other by means of a strong spring and are connected to the walls by weak springs. Simulate the behavior of (w_1, w_2) for t from 0 to 30 with steptime 0.01, using MATLAB, your SIMULINK block diagram and the same initial conditions and integration method as in task 5. Observe that the masses appear to oscillate in antiphase with relatively high frequency about 'equilibria' that are themselves slowly oscillating in phase with each other. Explain this behavior mathematically, and determine the slow and the fast frequencies. Hint: for the mathematical explanation you do not need to use a goniometric formula like the one from task 6.

Until now we assumed the springs in our mass spring system were ideal. We now drop this assumption, and consider the following modification of our mass spring system:



So, the middle spring is damped. We assume that the damping force is proportional to the velocity of the spring $\vec{v}$: $\vec{F}_{\text{damper}} = -d_2 \cdot \vec{v}$ with d_2 a constant > 0. We shall simulate the behavior of the modified system for different values of d_2.

Tasks

9. Derive the new force balances for both masses. Rewrite these two force balances, the same way as in task 2, into $\frac{dw}{dt} = A_{\text{new}} w$, with $w = [w_1, \frac{dw_1}{dt}, w_2, \frac{dw_2}{dt}]^{\text{T}}$ and A_{new} a 4×4-matrix with constant coefficients. Notice that to obtain A_{new} you only need to slightly adjust the matrix A.

10. Calculate the eigenvalues of the matrix A_{new}. MAPLE could serve as a useful tool. What is the difference between these eigenvalues and the eigenvalues of the matrix A which you calculated in task 3 in terms of stability?

11. Using your SIMULINK block diagram and MATLAB, simulate the behavior of (w_1, w_2) for both the case in which $k_1 = 36$ and $k_2 = 4$ and the case in which $k_1 = 4$ and $k_2 = 36$. For d_2 take $d_2 = 0.1$, $d_2 = 0.3$ and $d_2 = 1$. Let t run from 0 to 50 with step time 0.01. Again use 'rk45' as integration method. Take the same initial conditions as in the previous simulations. What do you see?

12. Explain this behavior both mathematically, using the eigenvalues from task 10, and physically.