5. Constrained nonlinear optimization

Version: 28-10-2015

Material (for details see )

- Chapter 12 in [FKS] (pp.283-320)

A reference e.g. L.12.2 refers to the corresponding Lemma in the book [FKS]

A pdf of the book: Faigle/Kern/Still, Algorithmic principles of Mathematical Programming. is to be found at: http://wwwwhome.math.utwente.nl/~stillgj/priv/
5.1 Introduction

We consider the minimization problem

\[
(P) \quad \min f(x) \quad \text{s.t.} \quad x \in \mathcal{F}
\]

where the feasible set is given by

\[
\mathcal{F} = \{ x \in \mathbb{R}^n \mid g_j(x) \leq 0, j \in J \}
\]

with \( J = \{1, \ldots, m\} \), \( f, g_j \in C^1(\mathbb{R}^n, \mathbb{R}) \)

(but no convexity assumptions).

Rem. We skip additional equality constraints \( h_i(x) = 0 \) to avoid technical difficulties.

Recall for \( \bar{x} \in \mathcal{F} \) the active index set \( J_{\bar{x}} \):

\[
J_{\bar{x}} = \{ j \in J \mid g_j(\bar{x}) = 0 \}
\]
Remark. Let $f \in C^1$ and $d \in \mathbb{R}^n$ be given. Then:

$$\nabla f(\bar{x})^T d < 0 \implies f(\bar{x} + td) < f(\bar{x}) \quad \text{for } t > 0, \ t \text{ small.}$$

Def. 5.1 A direction $d \in \mathbb{R}^n$ is called a strictly feasible direction at $\bar{x} \in \mathcal{F}$ if:

$$\nabla g_j(\bar{x})^T d < 0 \quad \forall j \in J_{\bar{x}} \quad (\star)$$

holds and a strictly feasible descent direction if in addition to $(\star)$:

$$\nabla f(\bar{x})^T d < 0$$
Th.5.2 [FKS, Th.12.4] [Necessary conditions]

Let $\bar{x} \in \mathcal{F}$ be a local minimizer of (P). Then the following equivalent conditions hold:

(a) (Primal condition) There does not exist a strictly feasible descent direction at $\bar{x}$.

(b) (Dual condition) There exists a non-trivial solution $\mu_0 \geq 0, \mu_j \geq 0, j \in J_{\bar{x}}$ (i.e., not all $\mu_j$'s are $= 0$) of:

$$\mu_0 \nabla f(\bar{x}) + \sum_{j \in J_{\bar{x}}} \mu_j \nabla g_j(\bar{x}) = 0 \quad (John \ condition)$$

We say: $\bar{x}$ is a John point.
Ex. 5.0  \([Farkas for LP]\)

Let \( c, a_i \in \mathbb{R}^n, i = 1, \ldots, m. \) Then precisely one of the following alternatives (I) or (II) are true:

(a)

(I): \( c^T x < 0, \ a_i^T x \leq 0, i = 1, \ldots, m \) has a solution \( x. \)

(II): there exist \( \mu_1 \geq 0, \ldots, \mu_m \geq 0 \) such that
\[
    c + \sum_{i=1}^{m} \mu_i a_i = 0
\]

(b)

(I): \( c^T x < 0, \ a_i^T x < 0, i = 1, \ldots, m \) has a solution \( x. \)

(II): there exist \( \mu_0 \geq 0, \mu_1 \geq 0, \ldots, \mu_m \geq 0, \) not all zero such that \( \mu_0 c + \sum_{i=1}^{m} \mu_i a_i = 0 \)

Hint. use: \( a^T x < 0 \) iff with some \( \xi < 0 \) we have \( a^T x - \xi \leq 0. \)
Under suitable conditions (constraint qualifications) the (stronger) KKT condition (i.e. the John-condition with \( \mu_0 = 1 \)) will be necessary for optimality.

**Def. 5.3** We say that the Mangasarian Fromovitz constraint qualification (MFCQ) holds at \( \bar{x} \in \mathcal{F} \), if there exists \( \hat{d} \) such that

\[
\nabla g_j(\bar{x})^T \hat{d} < 0, \quad \forall j \in J_{\bar{x}} \quad (\hat{d} \text{ is a strictly feasible direction})
\]

**Cor. 5.4** [FKS, Cor. 12.2] Let the MFCQ condition hold at \( \bar{x} \in \mathcal{F} \). If \( \bar{x} \) is a minimizer of (P) then the KKT condition holds (\( \bar{x} \) is a KKT point): there exist \( \mu_j \geq 0, \ j \in J_{\bar{x}} \), such that

\[
\nabla f(\bar{x}) + \sum_{j \in J_{\bar{x}}} \mu_j \nabla g_j(\bar{x}) = 0 \quad (\text{KKT-condition})
\]
**Rem.** Different from the convex case, in general nonlinear optimization, the KKT condition is not sufficient for minimality. Take as example:

\[
\min x^3 \quad \text{s.t.} \quad x \leq 0 \quad \text{with KKT-point } \overline{x} = 0 \text{ (no minimizer).}
\]

**Ex. 5.1** Consider (P) with convex \( g_j \in C^1 \forall j \). Show that the following are equivalent:

(a) There exists a point \( \overline{x} \in F \) satisfying MFCQ.

(b) There exists \( x^* \) such that \( g_j(x^*) < 0 \ \forall j \in J \) ("Slater condition").

(c) At all points \( \overline{x} \in F \), MFCQ holds.

**Def. 5.5** We say that the Linear independency constraint qualification (LICQ) holds at \( \overline{x} \in F \), if the active gradients

\[
\nabla g_j(\overline{x}), \ j \in J_{\overline{x}} \text{ are linearly independent.}
\]

**Ex. 5.2** [FKS, Ex. 12.11] Show that LICQ implies MFCQ.
First order sufficient conditions

Under additional assumptions (even in nonconvex optimization) the KKT condition can be sufficient for minimality.

**Def. 5.6** A point $\bar{x} \in \mathcal{F}$ is called a *strict local minimizer* of order $p = 1$ or $p = 2$ if with some $c > 0, \varepsilon > 0$:

$$f(x) - f(\bar{x}) \geq c\|x - \bar{x}\|^p \quad \forall x \in \mathcal{F}, \|x - \bar{x}\| < \varepsilon$$

**Th. 5.7** (see [FKS,Th.12.2]) (first order sufficient condition) Let the KKT condition hold for $\bar{x} \in \mathcal{F}$ with $\mu_j > 0, \ j \in J_{\bar{x}}$, and

$$|J_{\bar{x}}| = n, \ \nabla g_j(\bar{x}), \ j \in J_{\bar{x}} \ , \ \text{linearly independent.}$$

Then $\bar{x}$ is a strict local minimizer of (P) of order $p = 1$. 
Second order sufficient optimality conditions

In general (nonlinear opt.) for sufficient optimality conditions also second order information is needed.

**Def. 5.8** Let \( \bar{x} \in \mathcal{F} \). We introduce the set (of weakly feasible descent directions):

\[
C(\bar{x}) := \{ d \mid \nabla f(\bar{x})^T d \leq 0, \ \nabla g_j(\bar{x})^T d \leq 0, \ j \in J_{\bar{x}} \},
\]
called the cone of critical directions at \( \bar{x} \).

**Th. 5.9** [FKS, Th. 12.6] Let the KKT condition hold for \( \bar{x} \in \mathcal{F} \) with multipliers \( \mu_j \geq 0, j \in J_{\bar{x}} \), such that

\[
d^T \left[ \nabla^2 f(\bar{x}) + \sum_{j \in J_{\bar{x}}} \mu_j \nabla^2 g_j(\bar{x}) \right] d > 0 \quad \forall d \in C(\bar{x}) \setminus \{0\}.
\]

Then \( \bar{x} \) is a strict local minimizer of (P) of order \( p = 2 \).
Basic idea. This method is a generalization of the “steepest descent method” to constrained programs; it is based on “feasible descent directions”.

Consider again:

\[
(P) \quad \min_{x \in \mathcal{F}} f(x) , \quad \mathcal{F} = \{ x \mid g_j(x) \leq 0, \ j \in J \}
\]

Recall: \( d_k \) is a strictly feasible descent dir. in \( x_k \in \mathcal{F} \) if

\[
\nabla f(x_k)^T d_k < 0 , \ \nabla g_j(x_k)^T d_k < 0 , \ \forall j \in J_{x_k}
\]

Ex. 5.3 Let \( d_k \) be a strictly feasible descent direction in \( x_k \in \mathcal{F} \). Show that for any \( t > 0 \), small enough:

\[
f(x_k + td_k) < f(x_k) \text{ and } x_k + td_k \in \mathcal{F}
\]

Recall Theorem 5.2. There is no strictly feasible descent direction \( d_k \) at \( x_k \) iff \( x_k \) is a John point.
This observation leads to the following algorithm for computing *approximately* a John point $\overline{x}$ of (P).

### Feasible Direction Method (Zoutendijk)

**INIT:** Choose a starting point $x_0 \in \mathcal{F}$

**ITER:** WHILE $x_k$ is not a John point DO

**BEGIN**

Choose a strictly feasible descent direction $d_k$

Determine a solution $t_k$ for the problem

$$\min_{t > 0} \{ f(x_k + td_k) \mid x_k + td_k \in \mathcal{F} \}$$

Set $x_{k+1} = x_k + t_k d_k$.

**END**

**Question:** How to compute $d_k$?
Naive method: Solve the linear program,

\[
\begin{align*}
\min_{d,z} & \quad z \\
\text{s.t.} & \quad \nabla f(x_k)^T d - z \leq 0 \\
& \quad \nabla g_j(x_k)^T d - z \leq 0 \quad j \in J_{x_k} \\
& \quad \pm d_i \leq 1 \quad \forall i
\end{align*}
\]

(⋆)

where \(\pm d_i \leq 1\) is added to guarantee a finite optimal solution.

Remark. If the sol. \((z, d)\) in (⋆) satisfies \(z < 0\) then \(d\) is a strictly feasible descent direction and vice versa.

Remark. This “naive method” may result in jamming in the index sets \(J_{x_k}\), leading to “bad” convergence. Possibly \(x_k \to \bar{x}\) where \(\bar{x}\) is not a John point (see Wolfe’s example [FKS, p292].)
Better: The following method of Topkis and Veinott takes all constraints into account: compute \( d_k \) by solving

\[
\begin{align*}
\min_{d,z} & \quad z \\
\text{s.t.} & \quad \nabla f(x_k)^T d - z \leq 0 \\
& \quad \nabla g_j(x_k)^T d - z \leq -g_j(x_k) \quad \forall j \in J \\
& \quad \pm d_i \leq 1 \quad \forall i
\end{align*}
\]

**Ex. 5.4** \( LP_{x_k} \) has optimal value \( z = 0 \) precisely when \( x_k \) satisfies the John optimality conditions

*This method leads to a “positive” convergence result.*

**Th. 5.10** [FKS,Th.12.5] [Topkis and Veinott]
Assume that the Feasible Direction Method with \( d_k \) computed by \( LP_{x_k} \) generates the points \( x_\ell \). Suppose \( x_s \to \bar{x} \) for \( s \) from an infinite subset \( S \subset \mathbb{N} \).

Then \( \bar{x} \) is a John point.
**Convergence properties:** Roughly speaking, these feasible descent methods which only use first order (gradient) information show typically a (often slow) linear convergence similar to the steepest descent method in unconstrained optimization.
5.4 Penalty methods

Basic Idea: Transform the constrained problem

\[(P) \quad \min_{x \in \mathcal{F}} f(x), \quad \mathcal{F} = \{x \mid g_j(x) \leq 0, j \in J\}\]

into an unconstrained program.

**Penalty method:** Instead of (P) solve the unconstrained problem *(with suitable parameter \( r > 0; \) suitable \( p(x) \)):

\[(P_r) \quad \min_{x \in \mathbb{R}^n} p_r(x) = f(x) + r p(x)\]

Assumption: The feasible set \( \mathcal{F} \) is non-empty.

**Def. 5.12** \( p : \mathbb{R}^n \to \mathbb{R} \) is a *penalty function* with respect to \( \mathcal{F} \) if

\[ p(x) = 0 \quad \forall x \in \mathcal{F} \quad \text{and} \quad p(x) > 0 \quad \forall x \in \mathbb{R}^n \setminus \mathcal{F} \]
Examples. Setting \( g_j^+(x) = \max\{0, g_j(x)\} \) such penalty functions are given by:

\[
p(x) = \sum_{j \in J} g_j^+(x) \quad \text{or} \quad p(x) = \sum_{j \in J} (g_j^+(x))^2.
\]

Note. Typically \( g_j^+ \notin C^1 \) at \( \bar{x} \) with \( g_j(\bar{x}) = 0 \).

Ex. 5.5 \[FKS,\text{Ex.12.21}\] Let \( g : \mathbb{R}^n \to \mathbb{R} \) be a \( C^1 \)-function. Then also \( \hat{g}(x) = [g^+(x)]^2 \) is a \( C^1 \)-function. Determine \( \nabla \hat{g}(x) \).

Th. 5.12 \[FKS,\text{Th.12.9}\] \([\text{general convergence result}]\]
Let \( p(x) \) be a continuous penalty function wrt. \( \mathcal{F} \neq \emptyset \). Choose parameters \( r_k \to \infty \) and consider global minimizers \( x_k \) of the problems \( (P_{r_k}) \). If \( x_k \to x^* \) then \( x^* \) is a global minimizer of \( (P) \).
5.4.1 Exact penalty function

We consider the "exact" penalty problem \((P_r)\) (non-smooth)
\[
(P_r) \quad \min_{x \in \mathbb{R}^n} p_r(x) = f(x) + r \sum_{j \in J} g_j^+(x)
\]

The next theorem explains why this method is called "exact".

**Th. 5.13** [FKS, Th. 12.10]  Let (LICQ) be satisfied for the local minimizer \(\bar{x}\) of \((P)\) and assume that the sufficient optimality conditions of Th. 12.6 are fulfilled with multipliers \(\bar{\mu}_j > 0, \forall j \in J_x\). Then \(\bar{x}\) is also a ("exact") local minimizer of the penalty problem \((P_r)\) whenever
\[
r > \max\{\bar{\mu}_j \mid j \in J_x\}.
\]

**Ex. 5.6** (to illustrate the working of the different penalty functions)  Solve the program:
\[
\min x^2 \quad \text{s.t. } g(x) := 1 - x \leq 0
\]

by the penalty method with \(p(x) = (g^+(x))^2\) and by the exact penalty method with \(p(x) = g^+(x)\).
The proof of Th. 5.13 is based on the following “perturbation arguments”.

**Observation:** Under LICQ, a local minimizer $\bar{x}$ of (P) can be computed as a solution of *(necessary KKT conditions)*:

$$
\nabla f(x) + \sum_{j \in J_{\bar{x}}} \mu_j \nabla g_j(x) = 0
$$

\( (*) \)

$$
g_j(x) = 0, \ j \in J_{\bar{x}}
$$

For small perturbation vectors $0 \approx u \in \mathbb{R}^m$ we consider the *perturbed program*

$$(P_u) \quad \min_{x} f(x), \quad \text{s.t.} \quad g_j(x) \leq u_j, \ j \in J$$

with minimizer $x(u)$, minimal value $v(u)$ and perturbed KKT system:

$$
\nabla f(x) + \sum_{j \in J_{\bar{x}}} \mu_j \nabla g_j(x) = 0
$$

$$
g_j(x) = u_j, \ j \in J_{\bar{x}}
$$

depending on the parameter $u$
By applying the Inverse Function Theorem (under some regularity conditions) it can be shown that (locally around $\bar{u} = 0$) the functions $x(u)$ and $v(u)$ are $C^1$-functions of $u$.

**Th. 5.14** [FKS, Th. 12.8] Assume that (LICQ) is satisfied at the KKT-point $\bar{x} \in \mathcal{F}$ and that the sufficient optimality conditions of Th. 12.6 hold with multipliers $\bar{\mu}_j > 0$, $\forall j \in J_x$. Then there exists a neighborhood $U$ of $\bar{u} = 0$ and a continuously differentiable function $x : U \to \mathbb{R}^n$ such that $x(0) = \bar{x}$ and for all $u \in U$,

(i) $x(u)$ is a strict local minimizer for $(P_u)$.
(ii) $v(u) = f(x(u))$ satisfies $\nabla v(0) = -\bar{\mu}$
This method leads to algorithms for solving constrained programs that similarly to the Quasi-Newton methods in unconstrained optimization are superlinearly convergent.

To solve: \( \min_{x \in \mathcal{F}} f(x) , \mathcal{F} = \{ x \mid g_j(x) \leq 0, j \in J \} \)

Compute approximately a KKT point \( \bar{x} \) with Lagrangean multiplier \( \bar{\mu} \)

How? Better than in the primal method we try to compute a “feasible descent direction” \( d_k \) by solving a suitable quadratic subproblem (also using second order information).

To find \( d_k \), for given \( (x_k, \mu_k) \), possibly \( x_k \notin \mathcal{F} \), we solve the quadratic program

\[
(Q_k) \quad \min_d \quad \nabla f(x_k)^T d + \frac{1}{2} d^T L_k d \quad \text{s.t.} \\
\nabla g_j(x_k)^T d + g_j(x_k) \leq 0 \quad \forall j \in J
\]

where \( L_k = \nabla^2 x L(x_k, \mu_k) \) (or, an approximation thereof).
Rem. Note that \((Q_k)\) takes all inequality constraints into account, similarly to the primal method of Topkis-Veinott.

Now, computing a minimizer \(d_k\) of \((Q_k)\) we have to consider two cases: \(d_k = 0\) or \(d_k \neq 0\).

**Lem. 5.15 [FKS,Lem.12.4]** If \(d_k = 0\) is a minimizer of \((Q_k)\), with corresponding multiplier \(\mu_{k+1}\) then \(x_k\) is a KKT-point for \((P)\) with multiplier \(\mu_{k+1}\).

If \(d_k \neq 0\) is the minimizer of \((Q_k)\), we use \(d_k\) as search direction for the next iteration:

\[
x_{k+1} = x_k + t \, d_k.
\]

A problem appears: \(d_k \neq 0\) need not be a descent direction for \(f(x)\) (see [FKS, Ex.12.25]).

**Ex. 5.7** If \(x_k \in \mathcal{F}\) and \(d_k \neq 0\) is a solution of \((Q_k)\) with \(L_k \succ 0\) (pd.) then \(\nabla^T f(x_k) d_k < 0\), i.e., \(d_k\) is a descent direction for \(f\).
However \(d_k \neq 0\) is always a descent direction for the exact penalty function (for suitable \(r > 0\))

\[
p_r(x) = f(x) + r \sum_{j \in J} g_j^+(x)
\]

In this context, \(p_r\) is called a merit function

**Th. 5.16 [FKS, Th. 12.12]** Let the matrix \(L_k\) be positive definite and let \(d_k \neq 0\) a minimizer for \((Q_k)\) with corresponding multiplier \(\mu_{k+1}\). Then \(d_k\) is a descent direction for \(p_r\), i.e.,

\[
p_r(x_k + td_k) < p_r(x_k) \quad \text{for small} \quad t > 0,
\]

provided \(r \geq \max\{(\mu_{k+1})_j \mid j \in J\}\).
Under the hypothesis of Th. 5.14, a local minimizer of \((P)\) is also a local minimizer of the merit function \(p_r\). So Th. 5.16 suggests the following algorithm for computing a local minimizer of \(p_r(x)\) and hence a candidate minimizer of \((P)\).

**SQP-Method with merit function \(p_r\), \((\text{possibly } x_k \notin \mathcal{F})\)**

**INIT:** Choose \(x_0 \in \mathbb{R}^n\), \(r > 0\) large enough, \(L_0 \in \mathbb{R}^{n \times n}\) positive definite  

**ITER:** WHILE \(x_k\) is not a KKT-point DO  
BEGIN  
Compute a solution \(d_k\) of subproblem \((Q_k)\),  
Compute a solution \(t_k\) for the problem  
\((*) \quad \min_{t>0} p_r(x_k + td_k)\)  
Set \(x_{k+1} = x_k + t_k d_k\) and choose \(L_{k+1}\) positive definite.  
END
REMARK. Concerning the convergence properties of the SQP-method.

- For equality constrained programs, i.e., \( g_j(x) = 0 \), the solution \( d_k \) of \((Q_k)\) with \( L_k := \nabla^2_x L(x_k, \mu_k) \), coincides with the Newton iteration step at \( x_k, \mu_k \) applied to the KKT system (\(*\), p.18). So, under suitable assumptions the SQP method with (full step) \( x_{k+1} = x_k + d_k \) is locally quadratically convergent to a local minimizer \( \bar{x} \) of (P) (see [FKS, Ex.12.26]).

- Under suitable assumptions, if the SQP method generates \( t_k, d_k \) and \( x_{k+1} = x_k + t_k d_k \) such that

\[
t_k \to 1 \quad \text{and} \quad x_k \to \bar{x},
\]

then superlinear convergence occurs.
[The Maratos Effect:] Unfortunately, $t_k \to 1$ is not generally true for the line minimization $(\ast)$ of the SQP-algorithm. It may even happen for $x_k$ arbitrarily close to the solution $\bar{x}$ of (P) that the full Newton step $x_{k+1} = x_k + d_k$ increases the merit function to

$$p_r(x_k + d_k) > p_r(x_k)$$

(see [FKS, Ex.12.25] for an illustrative example). This negative effect, destroying the superlinear convergence, was discovered by Maratos. Fortunately, it is rarely observed in practice.