1. Let \( f : \mathbb{R}^m \to \mathbb{R} \) be a convex function \( f(y) \) on \( \mathbb{R}^m \) and let \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \) be given.

(a) Show that the function \( g(x) := f(Ax + b) \) is a convex function of \( x \) on \( \mathbb{R}^n \). [3 points]

(b) Suppose that \( f \) is strictly convex. Show that then \( g(x) := f(Ax + b) \) is strictly convex if and only if \( A \) has (full) rank \( n \).

Hint: Recall that \( f \) is strictly convex if for any \( y_1 \neq y_2, 0 < \lambda < 1 \) it holds:
\[
    f(\lambda y_1 + (1 - \lambda)y_2) < \lambda f(y_1) + (1 - \lambda)f(y_2).
\]

2. For given \( S \subset \mathbb{R}^n \) we define the convex hull \( \text{conv}(S) \) by
\[
    \text{conv}(S) = \left\{ x = \sum_{i=1}^{m} \lambda_i x_i \mid \sum_{i=1}^{m} \lambda_i = 1; \ x_i \in S, \lambda_i \geq 0 \ \forall i; \ m \in \mathbb{N} \right\}
\]
Show that \( \text{conv}(S) \) is the smallest convex set containing \( S \):

(a) Show that the set \( \text{conv}(S) \) is convex. [3 points]

(b) Show that for any convex set \( C \) containing \( S \) we must have \( \text{conv}(S) \subset C \).

(Hint: You may use without proof any Lemma, Theorem etc. from the course) [3 points]

3. Consider with \( 0 \neq c \in \mathbb{R}^n \) the program:
\[
(P) \quad \min_{x \in \mathbb{R}^n} \ c^T x \quad \text{s.t.} \quad x^T x \leq 1.
\]

(a) Show that \( \bar{x} = -\frac{c}{\|c\|} \) is the minimizer of \( (P) \) with minimum value \( v(P) = -\|c\| \). [2 points]

(\( \|x\| \) means here the Euclidian norm.)

(b) Compute the solution \( \bar{y} \) of the Lagrangian dual \( (D) \) of \( (P) \). Show in this way that for the optimal values strong duality holds, i.e., \( v(D) = v(P) \). [4 points]
4. Consider the problem (in connection with the design of a cylindrical can with height $h$, radius $r$ and volume at least $2\pi$ such that the total surface area is minimal):

$$\text{min } f(h,r) := 2\pi(r^2 + rh) \quad \text{s.t. } -\pi r^2 h \leq -2\pi, \quad (\text{and } h > 0, r > 0)$$

(a) Compute a (the) solution $(\overline{h}, \overline{r})$ of the KKT conditions of $(P)$. Show that $(P)$ is not a convex optimization problem. [4 points]

(b) Show that the solution $(\overline{h}, \overline{r})$ in (a) is a local minimizer. Why is it the unique global solution? [3 points]

Hint: Use the sufficient optimality conditions

5. Consider the closed set

$$\mathcal{K} = \{ x \in \mathbb{R}^2 \mid x_1 + 2x_2 \geq 0 \text{ and } 3x_1 + x_2 \geq 0 \}$$

(a) Prove that $\mathcal{K}$ is a convex cone. [2 points]

(b) Prove that $\mathcal{K}$ is full-dimensional. [1 point]

(c) Prove that $\mathcal{K}$ is pointed. [2 points]

(d) Find the dual cone to $\mathcal{K}$. [1 point]

6. We will consider bounds to the optimal value of the following problem:

$$\min_{x} \quad 5x_1^2 - 4x_1x_2 - 2x_1 + x_2^2 + 2$$

$$\text{s.t. } \quad x \in \mathbb{R}^2.$$  

(A)

(a) Give an upper bound on the optimal value of problem (A). [1 point]

(b) Formulate a sum-of-squares optimisation problem to give a lower bound on the optimal value of problem (A). [1 point]

(c) For fixed $u \in \mathbb{R}$, consider the polynomial $f(x) = 5x_1^2 - 4x_1x_2 - 2x_1 + x_2^2 + u$. [2 points]

Write the constraint that $f$ is a sum-of-squares polynomial explicitly as a positive semidefinite constraint.

7. (Automatic additional points) [4 points]

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A copy of the lecture-sheets may be used during the examination.
Good luck!