Ex.1.0 Model and solve Kepler’s problem (see sheets Chapter1)

Lemma 2.5 (Jensen’s inequality)
Let $f$ be a convex function defined on a convex set $C \subseteq \mathbb{R}^n$. Let the points $x^1, \cdots, x^k \in C$ be given and let $\lambda^1, \cdots, \lambda^k \geq 0$ be such that $\sum_{i=1}^{k} \lambda^i = 1$, $k \geq 2$. Then

$$\sum_{i=1}^{k} \lambda^i x^i \in C \quad \text{and} \quad f \left( \sum_{i=1}^{k} \lambda^i x^i \right) \leq \sum_{i=1}^{k} \lambda^i f(x^i).$$

Hint: proof by induction wrt. $k$.

Ex.2.7 (Max-value of a convex function is attained at an extreme point) Consider with compact, convex $\mathcal{F} \subseteq \mathbb{R}^n$ and convex, continuous function $f : \mathcal{F} \to \mathbb{R}$ the max problem:

$$(P) \quad \max f(x) \quad \text{s.t.} \quad x \in \mathcal{F}.$$ 

Show that the maximum value of (P) is attained (also) at an extreme point of $\mathcal{F}$. (Hint: Use the Krein-Milman Theorem)
**Ex. 2.10** Let $f$ be a twice continuously differentiable function on the open convex set $C$. Show:

\[ \nabla^2 f(x) \text{ positive definite } \forall x \in C \implies f \text{ is strictly convex.} \]

**Ex.3.1** *(Lemma 3.3)* Let $f : C \to \mathbb{R}$, $C \subset \mathbb{R}^n$ convex, be a convex function. Then a (strict) local minimizer of $f$ is a (strict) global minimizer.

**Ex.3.3** *(General characterization of minimizers)* Consider with convex $F \subset \mathbb{R}^n$ and convex $C^1$-function $f$ the problem: \( (P) \ \min f(x) \ \text{s.t.} \ \ x \in F. \)

Show that $\overline{x} \in F$ is a (global) minimizer of $(P)$ if and only if \[ \nabla f(\overline{x})^T (x - \overline{x}) \geq 0 \ \forall x \in F \] holds.

**Ex.3.4** Show that an ideal Slater point of $F$ is in the relative interior of $F$. 
Ex. 4.5. \((CO)\) \(\min_{x \in \mathbb{R}^2} e^{-x_2} \text{ s.t. } \sqrt{x_1^2 + x_2^2} \leq x_1\)

Show: The feasible set is \(\mathcal{F} = \{(x_1, x_2) \mid x_2 = 0, x_1 \geq 0\}\) (all feasible points are minimizers) and
\(\nu(WD) = -\infty < \nu(D) = 0 < \nu(CO) = 1\).

Ex. 5.0 [Farkas for LP]
Let \(\mathbf{c}, \mathbf{a}_i \in \mathbb{R}^n, i = 1, \ldots, m\). Then precisely one of the following alternatives (I) or (II) are true:

(a)
(I): \(\mathbf{c}^T \mathbf{x} < 0, \mathbf{a}_i^T \mathbf{x} \leq 0, i = 1, \ldots, m\) has a solution \(\mathbf{x}\).
(II): there exist \(\mu_1 \geq 0, \ldots, \mu_m \geq 0\) such that \(\mathbf{c} + \sum_{i=1}^m \mu_i \mathbf{a}_i = 0\)

(b)
(I): \(\mathbf{c}^T \mathbf{x} < 0, \mathbf{a}_i^T \mathbf{x} < 0, i = 1, \ldots, m\) has a solution \(\mathbf{x}\).
(II): there exist \(\mu_0 \geq 0, \mu_1 \geq 0, \ldots, \mu_m \geq 0\), not all zero such that \(\mu_0 \mathbf{c} + \sum_{i=1}^m \mu_i \mathbf{a}_i = 0\)

Hint. use: \(\mathbf{a}^T \mathbf{x} < 0\) iff with some \(\xi < 0\) we have \(\mathbf{a}^T \mathbf{x} - \xi \leq 0\).
**Ex. 5.6** (to illustrate the working of the different penalty functions) Solve the program:

\[
\min x^2 \quad \text{s.t.} \quad g(x) := 1 - x \leq 0
\]

by the penalty method with \( p(x) = (g^+(x))^2 \) and by the exact penalty method with \( p(x) = g^+(x) \).

**Ex. 5.7** If \( x_k \in F \) and \( d_k \neq 0 \) is a solution of (Q\(_k\)) with \( L_k \succ 0 \) (pd.) then \( \nabla^T f(x_k) d_k < 0 \), i.e., \( d_k \) is a descent direction for \( f \).

As Exercise: Prove Lem. 5.15 If \( d_k = 0 \) is a minimizer of (Q\(_k\)) \((L_k \text{ positive semidefinite})\), with corresponding multiplier \( \mu_{k+1} \), then \( x_k \) is a KKT-point for (P) with multiplier \( \mu_{k+1} \).

Recall: (Q\(_k\)) is the program:

\[
(Q_k) \quad \min_d \quad \nabla f(x_k)^T d + \frac{1}{2} d^T L_k d \quad \text{s.t.} \quad \nabla g_j(x_k)^T d + g_j(x_k) \leq 0 \quad \forall j \in J
\]