Examination: Mathematical Programming I  (158025)  
June 30, 2003,  13.30-16.30

Ex.1  Prove the following statements.

(a) Let \( A \in \mathbb{R}^{n \times m} \) be a given matrix \((m \leq n)\). Then \( A^T A \) is positive definite if and only if \( A \) has full rank \((m)\).

(b) For a symmetric \((n \times n)\)-matrix \( A \) the following holds: \( A \) is positive semidefinite if and only if all eigenvalues \( \lambda_j \) of \( A \) are non-negative \((\lambda_j \geq 0, \ j = 1, \ldots, n)\).

Ex.2  Consider the primal-dual pair of linear problems:

\[
\begin{align*}
(P) & \quad \max_{\mathbf{x} \in \mathbb{R}^n} \mathbf{c}^T \mathbf{x} \quad \text{s.t.} \quad \mathbf{Ax} \leq \mathbf{b} \\
(D) & \quad \min_{\mathbf{y} \in \mathbb{R}^m} \mathbf{b}^T \mathbf{y} \quad \text{s.t.} \quad \mathbf{A}^T \mathbf{y} = \mathbf{c}, \quad \mathbf{y} \geq 0
\end{align*}
\]

Show the following:

(a) There exists a feasible point for \( D \) (i.e. a point satisfying \( \mathbf{A}^T \mathbf{y} = \mathbf{c}, \ \mathbf{y} \geq \mathbf{0} \)) if and only if \( \mathbf{c}^T \mathbf{x} \leq 0 \) is implied by \( \mathbf{Ax} \leq \mathbf{0} \).

(b) Let the feasible set \( \mathcal{F}_P = \{ \mathbf{x} \mid \mathbf{Ax} \leq \mathbf{b} \} \) be non-empty. Show:

The value \( f(\mathbf{x}) = \mathbf{c}^T \mathbf{x} \) is bounded from above on \( \mathcal{F}_P \) if and only if \( \mathbf{c}^T \mathbf{x} \leq 0 \) is implied by \( \mathbf{Ax} \leq \mathbf{0} \).

Ex. 3

(a) Show that \( f(\mathbf{x}) = \|\mathbf{x}\| \) (\( \|\cdot\| \) any norm on \( \mathbb{R}^n \)) defines a convex function \( f : \mathbb{R}^n \to \mathbb{R} \).

(b) Let \( g : \mathbb{R}^n \to I, \ I \subset \mathbb{R} \) be convex and \( f : I \to \mathbb{R} \) be convex and non-decreasing. Show that the composition \( f \circ g(\mathbf{x}) = f(g(\mathbf{x})) \) of the functions \( f \) and \( g \) is convex.

(c) Show: The function \( f(\mathbf{x}) = e^{\|\mathbf{x}\|} \) is convex on \( \mathbb{R}^n \) (for any norm \( \|\mathbf{x}\| \) on \( \mathbb{R}^n \)).
Ex. 4

(a) Let \( f : (a, b) \rightarrow \mathbb{R} \) be a convex function. Show for all \( x \in (a, b) \):
\[
\partial f(x) = \{ d \in \mathbb{R} \mid f'_-(x) \leq d \leq f'_+(x) \}.
\]

(b) Determine the subdifferentials for the function \( f(x) = |x^2 - 1| \) at the points \( x_0 = 0 \), \( x_1 = 1 \), and \( x_2 = 4 \). (Is the function \( f \) convex?)

Ex. 5 Given the function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \),
\[ f(x) = x_1^3 + e^{3x_2} - 3x_1e^{x_2}. \]

(a) Find the critical points (i.e. points satisfying \( \nabla f(x) = 0^T, x = (x_1, x_2) \)) of the function \( f \) and determine the local minimizers.

(b) Does there exist a global minimizer or a global maximizer of \( f \) on \( \mathbb{R}^n \)?

(c) Suppose we apply the steepest descent method to \( f \). What can you say about the (local) convergence properties. (Quadratic or linear convergent? Give an estimate for the convergence factor.)

Points: 36+4 = 40

Ex. 1  a : 3 pt.
     b : 4 pt.
Ex. 2  a : 3 pt.
     b : 4 pt.
Ex. 3  a : 2 pt.
     b : 4 pt.
     c : 1 pt.
Ex. 4  a : 5 pt.
     b : 2 pt.
Ex. 5  a : 4 pt.
     b : 2 pt.
     c : 2 pt.

The script 'Mathematical Programming I' may be used during the examination. Good luck!