A linear process-algebraic format for probabilistic systems with data

Mark Timmer
June 25, 2010

Joint work with Joost-Pieter Katoen, Jaco van de Pol, and Mariëlle Stoelinga
Probabilistic Model Checking

Probabilistic model checking:
- Verifying quantitative properties,
- Using a probabilistic model
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- Non-deterministically choose one of the three transitions
- Probabilistically choose the next state
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![Diagram of a probabilistic system with transitions and probabilities]

- Non-deterministically choose one of the three transitions
- Probabilistically choose the next state

Limitations of previous approaches:
- Susceptible to the state space explosion problem
- Restricted treatment of data
Overview of our approach

Probabilistic specification (prCRL) → Instantiation → State space (PA) → Minimisation → Visualisation, Model checking
Overview of our approach

1. Probabilistic specification (prCRL)
2. Linearisation
3. Linear Probabilistic Process Equation (LPPE)
4. Instantiation
5. State space (PA)
6. Optimisation
   - Dead variables
   - Confluence
7. Visualisation
8. Model checking
Strong probabilistic bisimulation

Equivalent PAs: strong probabilistic bisimilar PAs
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Equivalent PAs: strong probabilistic bisimilar PAs

**Strong bisimulation**

An equivalence relation $R$ is a strong bisimulation if $(p, q) \in R$ and $p \xrightarrow{a} p'$ imply that $q \xrightarrow{a} q'$ such that $(p', q') \in R$. 

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2 A process algebra with data and probability: prCRL

3 Linear probabilistic process equations

4 Linearisation: from prCRL to LPPE

5 Case study: a leader election protocol

6 Conclusions and Future Work
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Specification language prCRL:

- Based on $\mu$CRL (so data), with additional probabilistic choice
- Semantics defined in terms of probabilistic automata
- Minimal set of operators to facilitate formal manipulation
- Syntactic sugar easily deflatable
A process algebra with data and probability: prCRL

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The grammar of prCRL process terms

Process terms in prCRL are obtained by the following grammar:

\[
p ::= Y(\vec{t}) \mid c \Rightarrow p \mid p + p \mid \sum_{x:D} p \mid a(\vec{t})\sum_{x:D} f : p
\]

Process equations and processes

A process equation is something of the form $X(\vec{g} : \vec{G}) = p$. 
An example specification

Sending an arbitrary natural number

\[ X(\text{active} : \text{Bool}) = \]
\[ \text{not(active)} \Rightarrow \text{ping} \cdot \sum_{b : \text{Bool}} X(b) \]
\[ + \text{active} \Rightarrow \tau \sum_{n : \mathbb{N} > 0} \frac{1}{2^n} : \left( \text{send}(n) \cdot X(\text{false}) \right) \]
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---

**Diagram:**

- **States:**
  - \( X(\text{false}) \)
  - \( \sum_{b : \text{Bool}} X(b) \)
  - \( \text{send}(1) \)
  - \( \text{send}(2) \)

- **Transitions:**
  - \( \text{ping} \) from \( X(\text{false}) \)
  - \( \text{ping} \) from \( \sum_{b : \text{Bool}} X(b) \)
  - \( \tau \)
  - \( 0.5 \)
  - \( 0.25 \)
  - \( \text{send}(1) \cdot X(\text{false}) \)
  - \( \text{send}(2) \cdot X(\text{false}) \)
  - \( \ldots \)
Compositionality using extended prCRL

For compositionality we introduce extended prCRL. It extends prCRL by parallel composition, encapsulation, hiding and renaming.
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\[
X(n : \{1, 2\}) = \text{write}_X(n) \cdot X(n) + \text{choose} \sum_{n' : \{1, 2\}} \frac{1}{2} : X(n')
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\[
Y(m : \{1, 2\}) = \text{write}_Y(m) \cdot Y(m) + \text{choose}' \sum_{m' : \{1, 2\}} \frac{1}{2} : Y(m')
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A linear format for prCRL: the LPPE

**LPPEs** are a subset of prCRL specifications:

\[
X(g : \tilde{G}) = \sum_{d_1 : \tilde{D}_1} c_1 \Rightarrow a_1(b_1) \sum_{e_1 : \tilde{E}_1} f_1 : X(n_1) \\
\ldots \\
+ \sum_{d_k : \tilde{D}_k} c_k \Rightarrow a_k(b_k) \sum_{e_k : \tilde{E}_k} f_k : X(n_k)
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Advantages of using LPPEs instead of prCRL specifications:

- Easy state space generation
- Straight-forward parallel composition
- Symbolic optimisations enabled at the language level
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- Easy state space generation
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- **Symbolic optimisations** enabled at the language level

**Theorem**

Every specification (without unguarded recursion) can be **linearised** to an LPPE, preserving strong probabilistic bisimulation.
Linear Probabilistic Process Equations – an example

Specification in prCRL

\[
X(\text{active} : \text{Bool}) = \\
\text{not(active)} \Rightarrow \text{ping} \cdot \sum_{b: \text{Bool}} X(b) \\
\text{+ active} \Rightarrow \tau \sum_{n: \mathbb{N} \geq 0} \frac{1}{2^n} : \text{send}(n) \cdot X(\text{false})
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\]

**Specification in LPPE**

\[
X(pc : \{1..3\}, n : \mathbb{N} \geq 0) = \\
\text{+ pc = 1 } \Rightarrow \text{ping} \cdot X(2, 1) \\
\text{+ pc = 2 } \Rightarrow \text{ping} \cdot X(2, 1) \\
\text{+ pc = 2 } \Rightarrow \tau \sum_{n: \mathbb{N} > 0} \frac{1}{2^n} : X(3, n) \\
\text{+ pc = 3 } \Rightarrow \text{send}(n) \cdot X(1, 1)
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Linearisation: a simple example without data

Consider the following prCRL specification:

\[ X = a \cdot b \cdot c \cdot X \]
Linearisation: a simple example without data

Consider the following prCRL specification:

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The control flow of \( X \) is given by:

```
\begin{array}{c}
1 \xrightarrow{a} 2 \\
2 \xrightarrow{b} 3 \\
3 \xrightarrow{c} 1 \\
\end{array}
```
Consider the following prCRL specification:

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The control flow of \( X \) is given by:

![Control flow diagram](image-url)
Consider the following prCRL specification:

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The control flow of \( X \) is given by:

The corresponding LPPE (initialised with \( \text{pc} = 1 \)):

\[
Y(\text{pc}: \{1, 2, 3\}) =
\]
\[
\begin{align*}
\text{pc} = 1 &\Rightarrow a \cdot Y(2) \\
+ \text{pc} = 2 &\Rightarrow b \cdot Y(3) \\
+ \text{pc} = 3 &\Rightarrow c \cdot Y(1)
\end{align*}
\]
Consider the following prCRL specification:

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\]

**Control flow:**

- 1
- 2
- 3
- 4

**LPPE:**

\[
Y(pc: \{1, 2, 3, 4\}, x: D) = \sum_{d:D} \begin{cases} 
    pc = 1 & \Rightarrow \text{get}(d) \cdot Y(2, d) \\
    pc = 2 & \Rightarrow \tau \cdot Y(3, x) \\
    pc = 2 & \Rightarrow \tau \cdot Y(4, x) \\
    pc = 3 & \Rightarrow \text{loss} \cdot Y(1, x) \\
    pc = 4 & \Rightarrow \text{send}(x) \cdot Y(1, x)
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Linearisation: a more complicated example with data

Consider the following prCRL specification:

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Control flow:

LPPE:

\[
Y(pc: \{1, 2, 3, 4\}, x: D) = \\
\sum_{d:D} pc = 1 \Rightarrow \text{get}(d) \cdot Y(2, d) \\
+ pc = 2 \Rightarrow \tau \cdot Y(3, x) \\
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+ pc = 3 \Rightarrow \text{loss} \cdot Y(1, x) \\
+ pc = 4 \Rightarrow \text{send}(x) \cdot Y(1, x)
\]

Initial process: \( Y(1, d_1) \).
Consider the following prCRL specification:

\[ X(d : D) = \sum_{e : D} a(d + e) \sum_{f : D} \frac{1}{|D|} \cdot \left( c(e) \cdot c(f) \cdot X(5) + c(e + f) \cdot X(5) \right) \]
Consider the following prCRL specification:

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X_1(d : D, e : D, f : D) = \sum_{e : D} a(d + e) \sum_{f : D} \frac{1}{|D|} : \left( c(e) \cdot c(f) \cdot X(5) + c(e + f) \cdot X(5) \right)
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\]

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X_1(d : D, e : D, f : D) = \sum_{e : D} a(d + e) \sum_{f : D} \frac{1}{|D|} : (c(e) \cdot c(f) \cdot X(5) + c(e + f) \cdot X(5))
\]

2. \[
X_1(d : D, e : D, f : D) = \sum_{e : D} a(d + e) \sum_{f : D} \frac{1}{|D|} \cdot X_2(d, e, f)
\]
   \[
X_2(d : D, e : D, f : D) = c(e) \cdot c(f) \cdot X(5) + c(e + f) \cdot X(5)
\]

3. \[
X_1(d : D, e : D, f : D) = \sum_{e : D} a(d + e) \sum_{f : D} \frac{1}{|D|} \cdot X_2(d, e, f)
\]
   \[
X_2(d : D, e : D, f : D) = c(e) \cdot X_3(d, e, f) + c(e + f) \cdot X_1(5, e, f)
\]
   \[
X_3(d : D, e : D, f : D) = c(f) \cdot X(5)
\]

4. \[
X_1(d : D, e : D, f : D) = \sum_{e : D} a(d + e) \sum_{f : D} \frac{1}{|D|} \cdot X_2(d, e, f)
\]
   \[
X_2(d : D, e : D, f : D) = c(e) \cdot X_3(d, e, f) + c(e + f) \cdot X_1(5, e, f)
\]
Consider the following prCRL specification:

\[
X(d : D) = \sum_{e:D} a(d+e) \sum_{f:D} \frac{1}{|D|} \cdot \left( c(e) \cdot c(f) \cdot X(5) + c(e + f) \cdot X(5) \right)
\]

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\]
\[X_3(d : D, e : D, f : D) = c(f) \cdot X_1(5, e, f)
\]
Linearisation: a more algorithmic approach

Consider the following prCRL specification:

\[
X(d : D) = \sum_{e : D} a(d + e) \cdot \left( \frac{1}{|D|} \cdot \left( c(e) \cdot c(f) \cdot X(5) + c(e + f) \cdot X(5) \right) \right)
\]

\[
\begin{align*}
4 & \quad X_1(d : D, e : D, f : D) = \sum_{e : D} a(d + e) \sum_{f : D} \frac{1}{|D|} \cdot X_2(d, e, f) \\
& \quad X_2(d : D, e : D, f : D) = c(e) \cdot X_3(d, e, f) + c(e + f) \cdot X_1(5, e, f) \\
& \quad X_3(d : D, e : D, f : D) = c(f) \cdot X_1(5, e, f)
\end{align*}
\]
Consider the following prCRL specification:

\[
X(d : D) = \sum_{e : D} a(d + e) \sum_{f : D} \frac{1}{|D|} : \left( c(e) \cdot c(f) \cdot X(5) + c(e + f) \cdot X(5) \right)
\]

4. \[
X_1(d : D, e : D, f : D) = \sum_{e : D} a(d + e) \sum_{f : D} \frac{1}{|D|} \cdot X_2(d, e, f)
\]
\[
X_2(d : D, e : D, f : D) = c(e) \cdot X_3(d, e, f) + c(e + f) \cdot X_1(5, e, f)
\]
\[
X_3(d : D, e : D, f : D) = c(f) \cdot X_1(5, e, f)
\]

\[
X(pc : \{1, 2, 3\}, d : D, e : D, f : D) =
\]

\[
pc = 1 \Rightarrow \sum_{e : D} a(d + e) \sum_{f : D} \frac{1}{|D|} \cdot X(2, d, e, f)
\]

\[
+ pc = 2 \Rightarrow c(e) \cdot X(3, d, e, f)
\]

\[
+ pc = 2 \Rightarrow c(e + f) \cdot X(1, 5, e, f)
\]

\[
+ pc = 3 \Rightarrow c(f) \cdot X(1, 5, e, f)
\]
In general, we always linearise in two steps:

1. Transform the specification to intermediate regular form (IRF) (every process is a summation of single-action terms)
2. Merge all processes into one big process by introducing a program counter

In the first step, **global parameters** are introduced to remember the values of bound variables.
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6 Conclusions and Future Work
Case study: a leader election protocol

- **Implementation** in Haskell:
  - Linearisation: from prCRL to LPPE
  - Parallel composition of LPPEs, hiding, renaming, encapsulation
  - Generation of the state space of an LPPE
  - Automatic constant elimination and summand simplification

- Manual **dead variable reduction**
Case study: a leader election protocol

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**Case study**

Leader election protocol à la Itai-Rodeh

- Two processes throw a **die**
  - *The process with the highest number will be leader*
  - *In case of a tie: throw again*
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Case study

Leader election protocol à la Itai-Rodeh

- Two processes throw a die
  - The process with the highest number will be leader
  - In case of a tie: throw again

- More precisely:
  - Passive thread: receive value of opponent
  - Active thread: roll, send, compare (or block)
A prCRL model of the leader election protocol

\[
P(id : \{\text{one, two}\}, \text{val} : \text{Die}, \text{set} : \text{Bool}) = \\
\quad \text{set} = \text{false} \Rightarrow \sum_{d : \text{Die}} \text{communicate}(id, \text{other}(id), d) \cdot P(id, d, \text{true}) \\
+ \sum_{d : \text{Die}} \text{communicate}(id, \text{other}(id), d) \cdot \text{checkValue}(\text{val}) \cdot P(id, \text{val}, \text{false}) \\
\]

\[
A(id : \{\text{one, two}\}) = \\
\quad \text{roll}(id) \sum_{d : \text{Die}} \frac{1}{6} \cdot \text{communicate}(\text{other}(id), id, d) \cdot \text{checkValue}(e) \cdot \\
( (d = e \Rightarrow A(id)) \\
+ (d > e \Rightarrow \text{leader}(id) \cdot A(id)) \\
+ (e > d \Rightarrow \text{follower}(id) \cdot A(id)) ) \\
\]

\[
C(id : \{\text{one, two}\}) = P(id, 1, \text{false}) \parallel A(id) \\
S = C(\text{one}) \parallel C(\text{two})
\]
Reductions on the leader election protocol model

In order to obtain reductions first linearise:

\[
\sum_{e21:Die} pc21 = 3 \land pc11 = 1 \land set11 \land val11 = e21 \Rightarrow
\]

\[
checkValue(val11) \sum multiply(1.0, 1.0):
\]

\[
(k1,k2):\{\ast\} \times \{\ast\}
\]

\[
Z(1, id11, val11, false, 1, 4, id21, d21, e21,
\]

\[
 pc12, id12, val12, set12, d12, pc22, id22, d22, e22)
\]
Reductions on the leader election protocol model

In order to obtain reductions first linearise:

\[ \sum_{e21:Die} \text{pc21} = 3 \land \text{pc11} = 1 \land \text{set11} \land \text{val11} = e21 \Rightarrow \]

\[ \text{checkValue(val11)} \sum (k_1,k_2):\{\ast\} \times \{\ast\} \]

\[ \text{multiply}(1.0, 1.0) : Z(1, id11, val11, false, 1, 4, id21, d21, e21, pc12, id12, val12, set12, d12, pc22, id22, d22, e22) \]

Before reductions:

- 18 parameters
- 14 summands
- 3763 states
- 6158 transitions
Reductions on the leader election protocol model

In order to obtain reductions first linearise:

\[
\begin{align*}
& \text{Before reductions:} \\
& \bullet 18 \text{ parameters} \\
& \bullet 14 \text{ summands} \\
& \bullet 3763 \text{ states} \\
& \bullet 6158 \text{ transitions} \\
& \text{After reductions:} \\
& \bullet 10 \text{ parameters} \\
& \bullet 12 \text{ summands}
\end{align*}
\]
Reductions on the leader election protocol model

In order to obtain reductions first linearise:

\[ pc21 = 3 \land \quad set11 \quad \Rightarrow \]

\[ \sum_{(k1,k2):\{\ast\} \times \{\ast\}} 1.0: \]

\[ \text{checkValue}(val11) \]

\[ Z(1, \text{false}, 4, d21, val11, val12, set12, pc22, d22, e22) \]

Before reductions:
- 18 parameters
- 14 summands
- 3763 states
- 6158 transitions

After reductions:
- 10 parameters
- 12 summands
- 1693 states (-55%)
- 2438 transitions (-60%)
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## Conclusions and Future Work

### Conclusions / Results

- We developed the **process algebra prCRL**, incorporating both **data** and **probability**.
- We defined a **normal form** for prCRL, the **LPPE**; starting point for symbolic optimisations and easy state space generation.
- We provided a **linearisation algorithm** to transform prCRL specifications to LPPEs, proved it **correct**, **implemented** it, and used it to show significant reductions on a **case study**.

---

**Future work**
- Develop additional reduction techniques, for instance confluence reduction (in progress).
- Generalise proof techniques such as cones and foci to the probabilistic case.
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- Develop additional reduction techniques, for instance confluence reduction (in progress).
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Questions